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Critical behaviour of a fluid in a random shear flow: renormalization group analysis of a simplified model

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Abstract

Critical behaviour of a fluid (binary mixture or liquid crystal), subjected to strongly anisotropic turbulent mixing, is studied by means of the field theoretic renormalization group. As a simplified model, relaxational stochastic dynamics of a non-conserved scalar order parameter, coupled to a random velocity field with prescribed statistics, is considered. The velocity is taken Gaussian, white in time, with a correlation function of the form $\propto\delta(t-t')/|\mathbf{k}_\perp|^{d+\xi}$, where $\mathbf{k}_\perp$ is the component of the wave vector, perpendicular to the distinguished direction ('direction of the flow')—the d-dimensional generalization of the ensemble introduced by Avellaneda and Majda (1990 *Commun. Math. Phys.* **131** 381) within the context of passive scalar advection. It is shown that, depending on the relation between the exponent ξ and the space dimensionality d, the system exhibits various types of large-scale self-similar behaviour, associated with different infrared attractive fixed points of the renormalization group equations. In addition to well-known asymptotic regimes (model A of equilibrium critical dynamics and a passively advected scalar with no self-interaction), the existence of a new, non-equilibrium and strongly anisotropic type of critical behaviour (universality class) is established, and the corresponding critical dimensions are calculated to the second order of the double expansion in ξ and $\varepsilon = 4 - d$ (two-loop approximation). The most realistic values of the model parameters (for example, $d = 3$ and the Kolmogorov exponent $\xi = 4/3$) belong to this class. The scaling behaviour appears anisotropic in the sense that the critical dimensions related to the directions parallel and perpendicular to the flow are essentially different. The results are in qualitative agreement with the results, obtained in experiments and simulations of fluid systems subjected to various kinds of regular and chaotic anisotropic flows.

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1. Introduction

Various systems of very different physical nature (ferromagnets and antiferromagnets, gas–vapour systems, binary liquid mixtures and alloys) reveal interesting singular behaviour when undergoing continuous (second-order) phase transition (that is, in the vicinity of their critical points). Specific heat, susceptibility, spontaneous magnetization, etc exhibit singular self-similar (power-law) behaviour, whose quantitative characteristics (critical dimensions and scaling functions) depend only on a few global characteristics of the system (like symmetry or space dimensionality). This universality is related to the existence in such systems of a wide range of strongly coupled degrees of freedom: it produces a kind of collective behaviour in which numerous irrelevant details of a specific system are wiped away. This classical subject is exposed in the monographs [1] and the literature cited therein.

Consistent qualitative and quantitative description of the equilibrium critical behaviour was achieved within the framework of the renormalization group (RG). In the RG approach, possible types of critical regimes (universality classes) are associated with infrared (IR) attractive fixed points of renormalizable field theoretic models. Most typical phase transitions belong to the universality class of the $O(N)$-symmetric φ^4 model of an N-component scalar order parameter (Landau–Ginzburg Hamiltonian). Universal characteristics of the critical behaviour depend only on N and the space dimensionality d and can be calculated in the form of the expansion in $\varepsilon = 4 - d$ or within other systematic perturbation schemes; see the monographs [2, 3] and the literature cited therein.

Dynamical critical behaviour (critical singularities of relaxation and correlation times, various kinetic and transport coefficients, etc) appears richer, less universal and is comparatively less understood. Different nature of the order parameter (conserved or non-conserved), inclusion of 'secondary' slow modes (densities of entropy or energy) and interaction with hydrodynamical degrees of freedom produce different types of critical dynamics for the same static model [3–5]. The reliable sets of second-order (two-loop) results were only recently fixed, and many important questions remain open; see the recent review paper [5] for discussion and bibliography.

It has long been realized that the behaviour of a real system near its critical point is extremely sensitive to external disturbances, geometry of the experimental setup, gravity, presence of impurities and so on. 'Ideal' equilibrium critical behaviour of an infinite system can be obscured by limited accuracy of measuring the temperature, finite-size effects, finite time of evolution (ageing) and so on. What is more, some disturbances (randomly distributed impurities in magnets and turbulent mixing of fluid systems) can produce completely new types of critical behaviour with rich and rather exotic properties (e.g., expansion in $\sqrt{\varepsilon}$ rather than in ε); see [6–8]. Over the past three decades, considerable attention has been attracted by the effects of various kinds of imposed flows (laminar shear flows, turbulent stirring and other types of deterministic or chaotic flows) on the behaviour of critical fluids, e.g., of binary liquid mixtures near the consolution point; see the papers [7–20] and references therein. This problem is closely related to another interesting issue: the effects of imposed flows on the dynamics of phase ordering—the growth of order through domain coarsening (spinodal decomposition), when a system is quenched from its high-temperature homogeneous phase into the low-temperature multi-phase coexistence region; see e.g. [15–19] and the literature cited therein.

Although very different, many of such systems exhibit a common interesting feature: existence of new non-equilibrium stationary states with (arguably) self-similar statistical properties and new sets of scaling exponents. Emergence of non-equilibrium steady states appears rather a generic and robust phenomenon, being observed in experiments, simulations

and analytic treatments of critical liquids with passive and active order parameters subjected to laminar or turbulent flows, various kinds of regular and chaotic synthetic velocity ensembles, cellular or shear flows, and so on [7–20]. In the presence of a distinguished direction, scaling behaviour of such systems appears strongly anisotropic, with different critical dimensions corresponding to different spatial directions [15–18].

The aim of the theory is to establish the existence of such regimes on the basis of microscopic dynamic models, to classify corresponding universality classes, to calculate their scaling dimensions within consistent approximations or regular perturbation schemes, to investigate their universality, dependence on the model parameters, and so on. In this paper, we will focus on the *anisotropic* turbulent mixing of critical fluids, because most real flows are strongly anisotropic; this anisotropy persists in the asymptotic critical regime and leads to new interesting effects.

The full-scale model of a critical fluid subjected to a strongly anisotropic turbulent stirring must deal with a conserved (binary mixtures) or non-conserved (liquid crystals) order parameter with mutual coupling with the velocity field, governed by nonlinear dynamic equations (e.g. the stochastic Navier–Stokes equation with an external random stirring force), and the anisotropy is introduced by the initial and/or boundary conditions. However, even for the equilibrium and isotropic case (model H in the traditional classification introduced in [4]), the consistent RG analysis of such a problem appears a most difficult task and has only recently been completed (see the discussion and references in [5] and sections 5.23–5.25 in book [3]), while theoretical description of fully developed turbulence on the basis of dynamic equations remains, in many respects, essentially an open problem.

In the present paper, we apply the field theoretic RG to a simplified 'minimal' model of a stirred critical fluid, which nevertheless appears rather nontrivial and captures the main property of the problem: existence of a new, non-equilibrium and strongly anisotropic, universality class of scaling behaviour. Namely, we consider a purely relaxational dynamics of a non-conserved passive scalar order parameter (model A in the terminology of [4]) coupled to the random velocity field with prescribed Gaussian statistics.

Recently, the models involving passive (no feedback on the velocity) linear (no self-interaction) scalar fields advected by such 'synthetic' velocity ensembles attracted enormous attention among the 'turbulent community' because of the insight they offer into the origin of intermittency and anomalous scaling in the real fluid turbulence; see the review paper [21] and references therein. In spite of their relative simplicity, such models reproduce many of the anomalous features of genuine turbulent heat or mass transport observed in experiments. Most popular is the Kazantsev–Kraichnan ensemble with the velocity correlation function of the form $\langle vv \rangle \propto \delta(t-t')k^{-d-\xi}$. Vanishing of the correlation time is necessary to ensure Galilean symmetry of the problem, while a power-law dependence on the wave number k mimics real self-similar properties of fully developed turbulence. For a *conserved* order parameter, it can be shown that the nonlinearity in the Navier–Stokes equation and a finite correlation time are indeed IR irrelevant (in the sense of Wilson) in the analysis of critical behaviour; see also the discussion in [8].

In the RG approach to the Kraichnan model and its descendants, reviewed in [22], the exponent ξ plays the role of a formal RG expansion parameter, analogous in this respect to the conventional $\varepsilon = 4 - d$.

Synthetic ensembles also allow one to easily introduce anisotropy, compressibility, etc and to study their effects on the behaviour of the scalar field. In this paper, we employ the d-dimensional generalization of a strongly anisotropic ensemble introduced in [23] in connection with the passive linear problem: the velocity field is oriented along a chosen direction $\mathbf{n}$ and its correlation function depends only on the coordinates perpendicular to $\mathbf{n}$; see also [24, 25].

In experiments, critical fluids with a non-conserved order parameter can be realized in twisted nematic liquid crystals; see the discussion and references in [17, 18]. In a wider context, such a model can be viewed as a model system for studying generic non-equilibrium phase transitions. Recently, significant progress has been made in classifying IR scaling behaviour of such phenomena, including driven diffusive systems, diffusion-limited reactions, growth, ageing and percolation processes, and so on; see e.g. [26–31] and references therein. Being analytically tractable, our model can serve as a possible testing ground in studying such scaling regimes and their universality within controlled approximations or a regular perturbative scheme.

Earlier, the field theoretic RG was applied to the stirred critical fluid in a number of studies. New types of critical behaviour were identified for model B in a Gaussian velocity ensemble [7, 8] and generalized model A with inclusion of a large-scale stirring force and the velocity field governed by the stochastic Navier–Stokes equation [14, 20], but only purely isotropic situations were considered. The RG ideas were also applied to the problem of phase separation and domain growth below the critical temperature; see e.g. [32, 33] and references therein. By contrast with critical phenomena, the RG approach to such problems suffers from the lack of an (obvious) small parameter (like $\varepsilon = 4 - d$ or ξ in our case) and should involve numerical (Monte Carlo) simulations [32] or additional phenomenological hypotheses [33].

The paper is organized as follows. In section 2, we give a detailed description of the model, present its field theoretic formulation and the corresponding diagrammatic technique. In section 3, we analyse canonical dimensions and ultraviolet (UV) divergences of the model. We show that, after an appropriate extension, the model becomes multiplicatively renormalizable and present the corresponding renormalized action functional. We also show that, for the extended model, independent canonical dimensions should be introduced for the directions parallel and perpendicular to the flow. In section 4, we derive the differential RG equations, introduce the RG functions (β functions and anomalous dimensions γ) and give the corresponding one-loop and two-loop expressions for the case of an N-component order parameter. In section 5, we analyse possible scaling regimes of the model, associated with the fixed points of the RG equations, and identify their ranges of stability in the ε–ξ plane. Three fixed points correspond to known regimes: free (Gaussian) regime, linear passive scalar and equilibrium model A. The fourth fixed point corresponds to a new, non-equilibrium and strongly anisotropic, universality class. For the most realistic values of the model parameters (namely, $d = 3$ and $0 < \xi < 2$; see section 2), it is the latter point that is IR attractive and governs the large-scale, long-time behaviour of the system. The corresponding critical dimensions depend on ε and ξ and can be systematically calculated as double series in those parameters; the explicit second-order results are presented in section 6. The scaling regime appears strongly anisotropic in the sense that the critical dimensions related to the directions parallel and perpendicular to the flow are different. Section 7 is reserved for discussion, comparison to the existing experimental and theoretical results and the conclusions.

In appendix A, we explore consequences of the Galilean symmetry for the renormalization of our model. The main points concerning the calculation of the renormalization constants and RG functions are briefly discussed in appendix B.

2. The model: field theoretic formulation

Relaxational dynamics of a non-conserved scalar order parameter $\varphi(x)$ with $x \equiv \{t, \mathbf{x}\}$ is described by a stochastic differential equation

$$\sigma_0 \partial_t \varphi(x) = -\delta \mathcal{H}(\varphi)/\delta\varphi(x) + f(x), \tag{2.1}$$

where $\partial_t = \partial/\partial t$, $\sigma_0 = 1/\Gamma_0$ is the reciprocal of the (constant) kinetic coefficient $\Gamma_0 > 0$ and $f(x)$ is a Gaussian random noise with zero mean and the pair correlation function

$$D_f(x - x') \equiv \langle f(x) f(x') \rangle = 2\sigma_0 \delta(t - t') \delta^{(d)}(\mathbf{x} - \mathbf{x}'), \tag{2.2}$$

d being the dimensionality of the $\mathbf{x}$ space. Near the critical point, the Hamiltonian $\mathcal{H}(\varphi)$ is taken in the Landau–Ginzburg form

$$\mathcal{H}(\varphi) = \int \mathrm{d}\mathbf{x} \left\{ -\frac{1}{2}\varphi(\mathbf{x})\partial^2 \varphi(\mathbf{x}) + \frac{\tau_0}{2}\varphi^2(\mathbf{x}) + \frac{\lambda_0}{4!}\varphi^4(\mathbf{x}) \right\}, \tag{2.3}$$

where $\partial_i = \partial/\partial x_i$ is the spatial derivative, $\partial^2 = \partial_i \partial_i$ is the Laplacian, $\tau_0 \propto (T - T_c)$ measures deviation from the critical temperature and $\lambda_0 > 0$ is the coupling constant; after the functional differentiation in (2.1) one has to replace $\varphi(\mathbf{x}) \to \varphi(x)$. The model (2.1)–(2.3) is referred to as model A [4]; its critical behaviour is very well understood [2–5].

Coupling with the velocity field $v_i(x)$ is introduced by the replacement

$$\partial_t \to \nabla_t = \partial_t + v_i \partial_i, \tag{2.4}$$

where ∇_t is the Lagrangian (Galilean covariant) derivative.

Let $\mathbf{n}$ be a unit constant vector that determines the distinguished direction ('direction of the flow'). Then any vector can be decomposed into the components perpendicular and parallel to the flow, for example, $\mathbf{x} = \mathbf{x}_\perp + \mathbf{n}x_\parallel$ with $\mathbf{x}_\perp \cdot \mathbf{n} = 0$. The velocity field will be taken in the form

$$\mathbf{v} = \mathbf{u} + \mathbf{n}v(t, \mathbf{x}_\perp), \tag{2.5}$$

where $\mathbf{u}$ is a constant vector parallel to $\mathbf{n}$ and $v(t, \mathbf{x}_\perp)$ is a scalar function independent of $x_\parallel$. Then the incompressibility condition is automatically satisfied:

$$\partial_i v_i = \partial_\parallel v(t, \mathbf{x}_\perp) = 0. \tag{2.6}$$

From now on, we set $\mathbf{u} = 0$ (the general case $\mathbf{u} \neq 0$ leads to no serious alterations in the RG analysis and will be briefly discussed at the end of section 7). For $v(t, \mathbf{x}_\perp)$ we assume a Gaussian distribution with zero mean and the pair correlation function of the form

$$\begin{aligned} \langle v(t, \mathbf{x}_\perp) v(t', \mathbf{x}'_\perp) \rangle &= \delta(t - t') \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} \exp\{\mathrm{i}\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')\} D_v(k) \\ &= \delta(t - t') \int \frac{\mathrm{d}\mathbf{k}_\perp}{(2\pi)^{d-1}} \exp\{\mathrm{i}\mathbf{k}_\perp \cdot (\mathbf{x}_\perp - \mathbf{x}'_\perp)\} \widetilde{D}_v(k_\perp), \quad k_\perp = |\mathbf{k}_\perp|, \end{aligned} \tag{2.7}$$

with the scalar coefficient functions of the form

$$D_v(k) = 2\pi \delta(k_\parallel) \widetilde{D}_v(k_\perp), \qquad \widetilde{D}_v(k_\perp) = D_0 k_\perp^{-d+1-\xi}. \tag{2.8}$$

Here $D_0 > 0$ is a constant amplitude factor and ξ an arbitrary exponent, which (along with the conventional $\varepsilon = 4 - d$) will play the part of a formal RG expansion parameter. The IR regularization in (2.7) is provided by the cut-off $k_\perp > m$ (by dimension, $\tau_0 \propto m^2$). (A precise form of the IR regularization is inessential; a sharp cut-off is the most convenient choice from the calculational viewpoints. Another possibility is to replace $k_\perp \to \sqrt{k_\perp^2 + m^2}$ in (2.8).) The natural interval for the exponent is $0 < \xi < 2$, when the so-called effective eddy diffusivity

$$\mathcal{V}(\mathbf{r}_\perp) = \int \frac{\mathrm{d}\mathbf{k}_\perp}{(2\pi)^{d-1}} \{1 - \exp(\mathrm{i}\mathbf{k}_\perp \cdot \mathbf{r}_\perp)\} \widetilde{D}_v(k_\perp) \tag{2.9}$$

has a finite limit for $m \to 0$; it includes the most realistic Kolmogorov value $\xi = 4/3$. The exponent ξ can also be viewed as a kind of Hölder exponent, which measures 'roughness' of the velocity field [21]; the 'Batchelor limit' $\xi \to 2$ corresponds to smooth velocity.

In order to ensure multiplicative renormalizability of the model, it is necessary to split the Laplacian in (2.3) into the parallel and perpendicular parts $\partial^2 \to \partial_\perp^2 + u_0\partial_\parallel^2$ by introducing a new parameter $u_0 > 0$ (in the anisotropic case, these two terms will be renormalized in a different way). Thus, equation (2.1) becomes

$$\sigma_0\nabla_t\varphi(x) = \partial_\perp^2\varphi(x) + u_0\partial_\parallel^2\varphi(x) - \tau_0\varphi(x) - \lambda_0\varphi^3(x)/6 + f(x); \tag{2.10}$$

this completes formulation of the model.

Interpretation of the splitting of the Laplacian term in (2.10) can be two-fold. On the one hand, the fluctuation models of the types (2.1) and (2.3) are phenomenological and, by construction, they must require all the IR relevant terms allowed by symmetry. The fact that the splitting is required by the renormalization procedure means that it is not forbidden by dimensionality or symmetry considerations and, therefore, it is natural to include the general value $u_0 \neq 1$ to the model from the very beginning. On the other hand, one can insist on studying the original model with $u_0 = 1$ and $SO(d)$ covariant Laplacian term, although that symmetry is broken to $SO(d-1)$ by the interaction with the anisotropic velocity ensemble. Then the extension of the model to the case $u_0 \neq 1$ can be viewed as a purely technical trick which is only needed to ensure the multiplicative renormalizability and to derive the RG equations. The latter should then be solved with the special initial data corresponding to $u_0 = 1$ (in renormalized variables this anyway will correspond to general initial data with $u \neq 1$). Since the IR attractive fixed point of the RG equations is unique for any given choice of the parameters ε and ξ (see section 5), the resulting IR behaviour will be the same as for the case of the extended model with general $u_0 \neq 1$.

According to the general theorem [34] (see also the monographs [2, 3]), our stochastic problem is equivalent to the field theoretic model of the extended set of fields $\Phi = \{\varphi', \varphi, \mathbf{v}\}$ with the action functional

$$\mathcal{S}(\Phi) = \sigma_0(\varphi')^2 + \varphi'\big[-\sigma_0\nabla_t\varphi + \partial_\perp^2\varphi + u_0\partial_\parallel^2\varphi - \tau_0\varphi - \lambda_0\varphi^3/6\big] + \mathcal{S}_v(\mathbf{v}). \tag{2.11}$$

The first few terms represent the De Dominicis–Janssen action functional for the stochastic problem (2.1), (2.2) at fixed $\mathbf{v}$; it involves an auxiliary scalar response field $\varphi'(x)$. All the required integrations over $x = \{t, \mathbf{x}\}$ and summations over the vector indices are implied, for example,

$$\varphi'\partial_\perp^2\varphi = \int \mathrm{d}t \int \mathrm{d}\mathbf{x}\, \varphi'(x)\partial_\perp^2\varphi(x).$$

It is worth noting that, owing to transversality of the velocity field (2.6), the derivative in the coupling term in (2.11) can also be moved onto the field φ' using integration by parts:

$$\begin{aligned}\varphi'(v_i\partial_i)\varphi &= \int \mathrm{d}t \int \mathrm{d}\mathbf{x}\, \varphi'(x)v(t,\mathbf{x}_\perp)\partial_\parallel\varphi(x)\\ &= -\int \mathrm{d}t \int \mathrm{d}\mathbf{x}(\partial_\parallel\varphi'(x))v(t,\mathbf{x}_\perp)\varphi(x).\end{aligned} \tag{2.12}$$

The last term in (2.11) corresponds to the Gaussian averaging over $\mathbf{v}$ with correlator (2.7) and has the form

$$\mathcal{S}_v(\mathbf{v}) = \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}\mathbf{x}_\perp\, \mathrm{d}\mathbf{x}'_\perp\, v(t,\mathbf{x}_\perp)\widetilde{D}_v^{-1}(\mathbf{x}_\perp - \mathbf{x}'_\perp)v(t,\mathbf{x}'_\perp), \tag{2.13}$$

where

$$\widetilde{D}_v^{-1}(\mathbf{r}_\perp) \propto D_0^{-1}r_\perp^{2(1-d)-\xi} \tag{2.14}$$

is the kernel of the inverse linear operation D_v^{-1} for the correlation function D_v in (2.8).

This formulation means that statistical averages of random quantities in the original stochastic problem coincide with the Green functions of the field theoretic model with action (2.11), given by functional averages with the weight $\exp \mathcal{S}(\Phi)$ (see equation (A.2)). This allows one to apply the field theoretic renormalization theory and renormalization group to our stochastic problem. The model (2.11) corresponds to a standard Feynman diagrammatic technique with three bare propagators (lines in the diagrams): $\langle \mathbf{vv} \rangle_0$, given by (2.7), (2.8), and the propagators of the scalar fields (in the frequency–momentum and time–momentum representations):

$$\langle \varphi \varphi' \rangle_0 = \langle \varphi' \varphi \rangle_0^* = \{-\mathrm{i}\sigma_0 \omega + \epsilon(\mathbf{k})\}^{-1} \leftrightarrow \theta(t-t')\sigma_0^{-1} \exp\{-\epsilon(\mathbf{k})(t-t')/\sigma_0\}, \tag{2.15}$$

where $\epsilon(\mathbf{k}) = \mathbf{k}_\perp^2 + u_0 \mathbf{k}_\parallel^2 + \tau_0$ and $\theta(\cdots)$ is the Heaviside step function, and

$$\langle \varphi \varphi \rangle_0 = 2\sigma_0 \left\{ \omega^2 \sigma_0^2 + \epsilon^2(\mathbf{k}) \right\}^{-1} \quad \leftrightarrow \quad \frac{1}{\epsilon(\mathbf{k})} \exp\{-\epsilon(\mathbf{k})|t-t'|/\sigma_0\}; \tag{2.16}$$

the propagator $\langle \varphi' \varphi' \rangle_0$ vanishes identically for any field theory of the type (2.11). The model also involves two types of vertices corresponding to the interaction terms $\varphi' \varphi^3$ and $\varphi'(v\partial_\parallel)\varphi$. The corresponding coupling constants ('charges') g_0 and w_0 defined are introduced by the relations

$$\lambda_0 = u_0^{1/2} g_0, \qquad D_0 = w_0 u_0 / \sigma_0, \tag{2.17}$$

so that by dimension $g_0 \sim \ell^{-\varepsilon}$ and $w_0 \sim \ell^{-\xi}$, where ℓ has the order of the smallest length scale of our problem. More precisely, these two lengths are rather different: the coupling g_0 in the Landau–Ginzburg model is conventionally related to the molecular length, while w_0 corresponds to the Kolmogorov (dissipation) scale of turbulence. However, in the following we will be interested in the behaviour of the correlation functions at distances much larger than both these lengths, which allows us not to distinguish them. Thus, we can write

$$g_0 \sim \Lambda^\varepsilon, \qquad w_0 \sim \Lambda^\xi, \tag{2.18}$$

where Λ sets the characteristic UV momentum scale.

By rescaling the fields, the coupling constant w_0 can be placed in front of the interaction term $\varphi'(v\partial)\varphi$ in the action (2.11), which is more familiar for the field theory. We do not do it, however, in order not to spoil the natural form of the covariant derivative, and assign the factor w_0 to the propagator $\langle \mathbf{vv} \rangle_0$.

3. Canonical dimensions and renormalization

It is well known that the analysis of UV divergences is based on the analysis of canonical dimensions ('power counting'); see e.g. [2, 3]. General dynamic models of the type (2.11), in contrast to static models (like e.g. (2.3)), have two scales: canonical dimension of some quantity F (a field or a parameter in the action functional) is completely characterized by two numbers, the frequency dimension d_F^ω and the momentum dimension d_F^k. They are determined such that $[F] \sim [T]^{-d_F^\omega} [L]^{-d_F^k}$, where L is the length scale and T is the time scale; see e.g. chapter 5 in book [3]. Our strongly anisotropic model, however, has two independent momentum scales, related to the directions perpendicular and parallel to the vector $\mathbf{n}$, and a more detailed specification of the canonical dimensions is necessary. Namely, one has to introduce two independent momentum canonical dimensions $d_F^\perp$ and $d_F^\parallel$ so that

$$[F] \sim [T]^{-d_F^\omega} [L_\perp]^{-d_F^\perp} [L_\parallel]^{-d_F^\parallel},$$

where $L_\perp$ and $L_\parallel$ are (independent) length scales in the corresponding subspaces. The dimensions are found from the obvious normalization conditions $d_{k_\perp}^\perp = -d_{\mathbf{x}_\perp}^\perp = 1, d_{k_\perp}^\parallel =$

Table 1. Canonical dimensions of the fields and parameters in the model (2.11).

F	φ	φ'	$\mathbf{v}$	σ, σ_0	u, u_0	m, μ, Λ	g_0	w_0	g, w
d_F^ω	0	1	1	-1	0	0	0	0	0
$d_F^\perp$	$(d-3)/2$	$(d-3)/2$	0	2	2	1	$4-d$	ξ	0
$d_F^\parallel$	$1/2$	$1/2$	-1	0	-2	0	0	0	0
$d_F^k = d_F^\perp + d_F^\parallel$	$d/2-1$	$d/2-1$	-1	2	0	1	$4-d$	ξ	0
$d_F = 2d_F^\omega + d_F^k$	$d/2-1$	$d/2+1$	1	0	0	1	$4-d$	ξ	0

$-d_{\mathbf{x}_\perp}^\parallel = 0, d_{k_\perp}^\omega = d_{k_\parallel}^\omega = 0, d_\omega^\omega = -d_t^\omega = 1$, and so on, and from the requirement that each term of the action functional (2.11) be dimensionless (with respect to all the three independent dimensions separately). The original momentum dimension can be found from the relation $d_F^k = d_F^\perp + d_F^\parallel$. Then, based on d_F^k and d_F^ω, one can introduce the total canonical dimension $d_F = d_F^k + 2d_F^\omega = d_F^\perp + d_F^\parallel + 2d_F^\omega$ (in the free theory, $\partial_t \propto \partial_\perp^2 \propto \partial_\parallel^2$), which plays in the theory of renormalization of dynamic models the same role as the conventional (momentum) dimension does in static problems; cf chapter 5 in book [3].

The full set of independent canonical dimensions is needed, in particular, to identify the completely dimensionless parameters, which can only appear as arguments in the renormalization constants and RG functions. Of course, the existence of several independent spatial scales is not too exotic; it was encountered in a number of models: ferroelectrics [35] (see also section 1.17 of book [3]), ferromagnets with anisotropically correlated quenched impurities [36, 37], continuous models of self-organized criticality [38], anisotropic versions of the Kardar–Parisi–Zhang equation [39], m-axial Lifshits points [40] and growing surfaces, driven by obliquely incident particle beams [31].

The canonical dimensions for the model (2.11) are summarized in table 1, including renormalized parameters, which will be introduced later on. From table 1 or, equivalently, from the relations (2.18), it follows that the model is logarithmic (the coupling constants g_0 and w_0 are simultaneously dimensionless) at $d = 4$ and $\xi = 0$, so that the UV divergences in the correlation functions manifest themselves as poles in $\varepsilon \equiv 4 - d, \xi$ and their linear combinations or, in general, as singularities at ε and $\xi \to 0$.

The total canonical dimension of an arbitrary 1-irreducible Green function $\Gamma = \langle \Phi \cdots \Phi \rangle_{1\text{-ir}}$ is given by the relation

$$d_\Gamma = d + 2 - \sum_\Phi N_\Phi d_\Phi, \qquad \sum_\Phi N_\Phi d_\Phi = N_{\varphi'} d_{\varphi'} + N_\varphi d_\varphi + N_\mathbf{v} d_\mathbf{v}. \tag{3.1}$$

Here $N_\Phi = \{N_\varphi, N_{\varphi'}, N_\mathbf{v}\}$ are the numbers of corresponding fields entering into the function Γ, and the summation over all types of the fields in (3.1) and analogous formulae below is always implied.

The total dimension d_Γ in logarithmic theory (that is, at $\varepsilon = \xi = 0$) is the formal index of the UV divergence $\delta_\Gamma = d_\Gamma|_{\varepsilon=\xi=0}$. Superficial UV divergences, whose removal requires counterterms, can be present only in those functions Γ for which δ_Γ is a non-negative integer. The counterterms are local, that is, in the frequency–momentum representation the counterterm to a given function Γ is a polynomial in $\omega, \mathbf{k}_\perp$ and $\mathbf{k}_\parallel$. Since the parameters u_0 and σ_0 are dimensionless with respect to the total dimension d_F, the index δ_Γ gives the degree of that polynomial (with the assumption that $\omega \sim k_\perp^2 \sim k_\parallel^2$); a detailed structure of the counterterms and their dependence on σ_0 and u_0 is directly found from the corresponding partial dimensions d_Γ^ω and $d_\Gamma^{\perp,\parallel}$. From table 1 and (3.1) we find

$$\delta_\Gamma = 6 - 3N_{\varphi'} - N_\varphi - N_v. \tag{3.2}$$

Dimensional considerations should be augmented by the observation that all the 1-irreducible functions without the field φ' (in particular, all functions involving only velocity fields) contain closed circuits of retarded propagators $\langle\varphi'\varphi\rangle_0$, vanish and do not require counterterms; see e.g. [3]. The action (2.11) is even with respect to the reflection $\varphi' \to -\varphi', \varphi \to -\varphi$, so that all correlation functions with odd total number of the fields φ' and φ also vanish (no diagrams for such functions can be constructed). It is therefore sufficient to consider only 1-irreducible functions with $N_{\varphi'} \geqslant 1$ and even sum $N_{\varphi'} + N_{\varphi}$. Straightforward analysis of expression (3.2) then shows that superficial UV divergences can be present only in the following 1-irreducible functions:

$$\begin{aligned}
&\langle\varphi'\varphi'\rangle \quad (\delta = 0) \quad \text{with the counterterm} \quad \varphi'\varphi', \\
&\langle\varphi'\varphi\rangle \quad (\delta = 2) \quad \text{with the counterterms} \quad \varphi'\partial_t\varphi, \quad \varphi'\partial_\parallel^2\varphi, \quad \varphi'\partial_\perp^2\varphi, \quad \tau_0\varphi'\varphi, \\
&\langle\varphi'\varphi^3\rangle \quad (\delta = 0) \quad \text{with the counterterm} \quad \varphi'\varphi^3, \\
&\langle\varphi'\varphi v\rangle \quad (\delta = 1),
\end{aligned}$$

for which the counterterm necessarily reduces to the form $\varphi'(v_i\partial_i)\varphi = \varphi' v\partial_\parallel\varphi$. All such terms are present in the action (2.11), so that our model appears multiplicatively renormalizable.

The superficial divergence in the function $\langle\varphi'\varphi v v\rangle$ with $\delta = 0$ and the counterterm $\varphi'\varphi v^2$, allowed by the dimension, is in fact forbidden by the Galilean symmetry. Furthermore, the latter requires that the counterterms $\varphi'\partial_t\varphi$ and $\varphi'(v_i\partial_i)\varphi$ enter the renormalized action only in the form of Lagrangian derivative $\varphi'\nabla_t\varphi$.

The arguments based on the Galilean symmetry are usually applied to the velocity field governed by the Navier–Stokes equation, and generally become invalid for synthetic Gaussian velocity ensembles. It turns out, however, that for a Gaussian ensemble with *vanishing* correlation time the Galilean symmetry takes place; see e.g. [21]. This issue, along with the consequences of the Galilean invariance for the renormalization in our model, is discussed in appendix A in detail.

We conclude that the renormalized action can be written in the form

$$\mathcal{S}_R(\varphi', \varphi) = \mathcal{S}_v(\mathbf{v}) + Z_1\sigma(\varphi')^2 + \varphi'\big[-Z_2\sigma\nabla_t\varphi + Z_3\partial_\perp^2\varphi + Z_4 u\partial_\parallel^2\varphi - Z_5\tau\varphi - Z_6 g u^{1/2}\mu^\varepsilon\varphi^3/6\big]. \tag{3.3}$$

Here σ, τ, u, w and g are renormalized analogues of the bare parameters (with the subscripts '0') and μ is the reference mass scale (additional arbitrary parameter of the renormalized theory). Since the first term $\mathcal{S}_v(\mathbf{v})$ is not renormalized, the amplitude D_0 is expressed in renormalized parameters as

$$D_0 = w_0 u_0/\sigma_0 = w u\mu^\xi/\sigma. \tag{3.4}$$

Expression (3.3) is equivalent to the multiplicative renormalization of the fields $\varphi \to \varphi Z_\varphi, \varphi' \to \varphi' Z_{\varphi'}$ and the parameters

$$\sigma_0 = \sigma Z_\sigma, \qquad \tau_0 = \tau Z_\tau, \qquad u_0 = u Z_u, \qquad g_0 = g\mu^\varepsilon Z_g, \qquad w_0 = w\mu^\xi Z_w \tag{3.5}$$

(no renormalization of the velocity field is needed: $Z_v = 1$). The constants in equations (3.3) and (3.5) are related as follows:

$$\begin{aligned}
&Z_1 = Z_\sigma Z_{\varphi'}^2, \qquad Z_2 = Z_\sigma Z_{\varphi'} Z_\varphi, \qquad Z_3 = Z_{\varphi'} Z_\varphi, \\
&Z_4 = Z_u Z_{\varphi'} Z_\varphi, \qquad Z_5 = Z_\tau Z_{\varphi'} Z_\varphi, \qquad Z_6 = Z_g Z_u^{1/2} Z_{\varphi'} Z_\varphi^3,
\end{aligned} \tag{3.6}$$

and from relation (3.4) one obtains

$$Z_u Z_w Z_\sigma^{-1} = 1. \tag{3.7}$$

The renormalization constants must be chosen to capture all the divergences at $\varepsilon, \xi \to 0$, so that the correlation functions of the renormalized model (3.3) have finite limits for $\varepsilon, \xi = 0$ when expressed in renormalized parameters σ, μ and so on. This requirement determines Z's up to regular parts, which are specified by the choice of the renormalization scheme. In practical calculations (see appendix B for the details) we use the minimal subtraction (MS) scheme, which is most convenient from the calculational viewpoints. However, one important remark is in order here. In models with a single UV regulator (say, model A with ε), the UV singularities manifest themselves as poles in ε, and the MS scheme is defined such that all the renormalization constants have the forms $Z = 1 +$ only poles in ε. In models with several regulators, like ε and ξ in our case, there are subtleties in defining the renormalization scheme: for example, the ratio ε/ξ can be treated as a pole or a finite quantity. According to the general theory of analytic regularization [41], the constants Z can be chosen such that the renormalized correlation functions appear analytic in the full set of regulators; in this scheme, the quantities like ε/ξ are treated as singularities and are included into Z's.

Another possibility, which we adopt in our calculation and refer to as the MS scheme, is to consider ε and ξ as small parameters of the same order and, therefore, to treat their ratios as non-singular and to include into Z's only poles in ε, ξ and all their linear combinations $(k\varepsilon + p\xi)$ with integers k and p (the latter indeed appear in higher order diagrams). In other words, one can set $\xi = a\varepsilon$ and arrive at the conventional situation with the single UV regulator ε and the unambiguously defined MS scheme, in which the renormalization constants have the forms '1+ only poles in ε'. In addition to the two dimensionless parameters—renormalized coupling constants g and w, the coefficients in Z's depend on the ratio $a = \xi/\varepsilon$ and can be singular for some special values of a (e.g. for $a = 0$).

Although the intermediate quantities (renormalization constants and RG functions) explicitly depend on the choice of the renormalization scheme, the final physical results (existence and stability of the fixed points and the values of the corresponding critical exponents) must be the same. These general statements were confirmed in [42] by the practical calculation in an analogous two-regulator model (namely, for the stochastic Navier–Stokes equation near two dimensions). There, the renormalization constants and RG functions (β functions and anomalous dimensions) were calculated explicitly in the two different renormalization schemes, described above, to the two-loop order.

It remains to note that in the *leading-order* approximation all these subtleties appear not too important in our model, and the independence of the results on the renormalization scheme can readily be verified from the explicit expressions for the diagrams; see the remark below equation (B.5).

4. RG functions and RG equations

Let us recall an elementary derivation of the RG equations; a detailed discussion can be found in the monographs [2, 3]. The RG equations are written for the renormalized correlation functions $G_R = \langle \Phi \cdots \Phi \rangle_R$, which differ from the original (unrenormalized) ones $G = \langle \Phi \cdots \Phi \rangle$ only by normalization and choice of parameters, and therefore can equally be used for analysing the critical behaviour. The relation $\mathcal{S}_R(\Phi, e, \mu) = \mathcal{S}(\Phi, e_0)$ between the functionals (2.11) and (3.3) results in the relations

$$G(e_0, \ldots) = Z_\varphi^{N_\varphi} Z_{\varphi'}^{N_{\varphi'}} G_R(e, \mu, \ldots) \tag{4.1}$$

between the correlation functions. Here, as usual, N_φ and $N_{\varphi'}$ are the numbers of corresponding fields entering into Γ (we recall that in our model $Z_v = 1$); $e_0 = \{\sigma_0, \tau_0, u_0, w_0, g_0\}$ is the full

set of bare parameters and $e = \{\sigma, \tau, u, w, g\}$ are their renormalized counterparts; the ellipsis stands for the other arguments (times, coordinates, momenta, etc).

We use $\widetilde{\mathcal{D}}_\mu$ to denote the differential operation $\mu\partial_\mu$ for fixed e_0 and operate on both sides of equation (4.1) with it. This gives the basic RG differential equation

$$\{\mathcal{D}_{\mathrm{RG}} + N_\varphi\gamma_\varphi + N_{\varphi'}\gamma_{\varphi'}\}G^R(e, \mu, \ldots) = 0, \tag{4.2}$$

where $\mathcal{D}_{\mathrm{RG}}$ is the operation $\widetilde{\mathcal{D}}_\mu$ expressed in the renormalized variables:

$$\mathcal{D}_{\mathrm{RG}} \equiv \mathcal{D}_\mu + \beta_g\partial_g + \beta_w\partial_w - \gamma_u\mathcal{D}_u - \gamma_\sigma\mathcal{D}_\sigma - \gamma_\tau\mathcal{D}_\tau. \tag{4.3}$$

Here we have written $\mathcal{D}_x \equiv x\partial_x$ for any variable x, and the anomalous dimensions γ are defined as

$$\gamma_F \equiv \widetilde{\mathcal{D}}_\mu \ln Z_F \quad \text{for any quantity } F, \tag{4.4}$$

and the β functions for the two dimensionless couplings g and w are

$$\beta_g \equiv \widetilde{\mathcal{D}}_\mu g = g[-\varepsilon - \gamma_g], \qquad \beta_w \equiv \widetilde{\mathcal{D}}_\mu w = w[-\xi - \gamma_w], \tag{4.5}$$

where the second equalities come from the definitions and relations (3.5).

Equations (3.6) result in the following relations between the anomalous dimensions:

$$\begin{aligned} &\gamma_1 = \gamma_\sigma + 2\gamma_{\varphi'}, \qquad \gamma_2 = \gamma_\sigma + \gamma_{\varphi'} + \gamma_\varphi, \qquad \gamma_3 = \gamma_{\varphi'} + \gamma_\varphi, \\ &\gamma_4 = \gamma_u + \gamma_3, \qquad \gamma_5 = \gamma_\tau + \gamma_3, \qquad \gamma_6 = \gamma_g + \gamma_u/2 + \gamma_{\varphi'} + 3\gamma_\varphi, \end{aligned} \tag{4.6}$$

while from (3.7) one obtains

$$\gamma_u + \gamma_w - \gamma_\sigma = 0. \tag{4.7}$$

The dimensions $\gamma_1 - \gamma_6$ are calculated from the corresponding renormalization constants using definition (4.4), while the RG functions entering equation (4.3) are easily found from relations (4.6) and (4.7):

$$\begin{aligned} &2\gamma_{\varphi'} = \gamma_1 - \gamma_2 + \gamma_3, \qquad 2\gamma_\varphi = \gamma_3 - \gamma_1 + \gamma_2, \\ &\gamma_u = \gamma_4 - \gamma_3, \qquad \gamma_\tau = \gamma_5 - \gamma_3, \qquad \gamma_\sigma = \gamma_2 - \gamma_3, \\ &\gamma_w = \gamma_2 - \gamma_4, \qquad \gamma_g = \gamma_1 - \gamma_2 - 3\gamma_3/2 - \gamma_4/2 + \gamma_6. \end{aligned} \tag{4.8}$$

The diagrams needed for our second-order calculation of the critical dimensions are presented in appendix B. One can see that the leading contributions to different renormalization constants (and hence to the corresponding anomalous dimensions) are of different order: $Z_{1,2,3} = 1 + O(g^2)$, $Z_{5,6} = 1 + O(g)$, $Z_4 = 1 + O(w)$. Practical calculation of the renormalization constants and anomalous dimensions are discussed in appendix B, and here we only present the leading-order results for the dimensions (4.6):

$$\begin{aligned} &\gamma_1 = \gamma_2 = b\kappa_1\tilde{g}^2 + O(\tilde{g}^3), \qquad \gamma_3 = \kappa_1\tilde{g}^2/6 + O(\tilde{g}^3), \qquad \gamma_6 = -3\kappa_2\tilde{g} + O(\tilde{g}^2), \\ &\gamma_4 = \tilde{w} + \kappa_1\tilde{g}^2/6 + O(\tilde{g}^3), \qquad \gamma_5 = -\kappa_1\tilde{g} + O(\tilde{g}^2), \end{aligned} \tag{4.9}$$

where we have denoted $\tilde{g} = g/(16\pi^2)$, $\tilde{w} = w/(4\pi^2)$ and $b = \ln(4/3) \approx 0.287\,683$; in counting the orders it is assumed that $w = O(g)$. For generality, we give the results for the $O(N)$-symmetric model with an N-component order parameter in (2.3); the additional symmetry factors are $\kappa_1 = (N+2)/3$ and $\kappa_2 = (N+8)/9$. In the following, we will only give the results for $N = 1$ and denote the new couplings $\tilde{g}, \tilde{w}$ simply by g, w. Then expressions (4.9) take on the form

$$\begin{aligned} &\gamma_1 = \gamma_2 = bg^2 + O(g^3), \qquad \gamma_3 = g^2/6 + O(g^3), \qquad \gamma_6 = -3g + O(g^2), \\ &\gamma_4 = w + g^2/6 + O(g^3), \qquad \gamma_5 = -g + O(g^2). \end{aligned} \tag{4.10}$$

5. Fixed points and scaling regimes

It is well known that possible large-scale scaling regimes of a renormalizable model are associated with IR attractive fixed points of the corresponding RG equations. In our model, the coordinates g_*, w_* of the fixed points are found from the equations

$$\beta_g(g_*, w_*) = 0, \qquad \beta_w(g_*, w_*) = 0, \tag{5.1}$$

with the β functions given in (4.5). The type of a fixed point is determined by the matrix

$$\Omega = \{\Omega_{ij} = \partial\beta_i/\partial g_j\}, \tag{5.2}$$

where β_i denotes the full set of the β functions and $g_j = \{g, w\}$ is the full set of couplings. For IR stable fixed points the matrix Ω is positive, i.e., the real parts of all its eigenvalues are positive.

From definitions (4.5), relations (4.8) and explicit expressions (4.10) for the anomalous dimensions, we derive the following leading-order expressions for the β functions:

$$\beta_g = g[-\varepsilon + 3g + w/2], \qquad \beta_w = w[-\xi + w] \tag{5.3}$$

with the corrections in the square brackets of order $O(g^2)$ and higher. From equations (5.1) and (5.3) we can identify four different fixed points; the matrix Ω appears triangular for all of them, so that its eigenvalues are simply given by the diagonal elements $\Omega_g = \partial\beta_g/\partial g$ and $\Omega_w = \partial\beta_w/\partial w$.

1. Gaussian (free) fixed point: $g_* = w_* = 0$; $\Omega_g = -\varepsilon$, $\Omega_w = -\xi$.

2. $w_* = 0$ (exact result to all orders), $g_* = \varepsilon/3$; $\Omega_g = \varepsilon$, $\Omega_w = -\xi$. In this regime, effects of the velocity field are irrelevant, the isotropy violated by the velocity ensemble is restored and the leading terms of the IR behaviour coincide exactly with those of the equilibrium model A. In particular, the basic critical dimensions do not depend on ξ and coincide to all orders in ε with the well-known static exponents η, ν for the Landau–Ginzburg model (2.3) and the dynamic exponent z for model A (see e.g. [3, 5]). However, corrections to the leading-order asymptotic expressions will be anisotropic and different from those for model A; in particular, the dependence on ξ will appear e.g. due to the correction exponent Ω_w.

3. $g_* = 0$ (exact result to all orders), $w_* = \xi$; $\Omega_g = (-\varepsilon + \xi/2)$, $\Omega_w = \xi$. In this regime, the nonlinearity φ^3 in the stochastic equation (2.1) becomes irrelevant, and we arrive at the model of a linear convection–diffusion equation for a passive scalar field φ. For the strongly anisotropic Gaussian velocity ensembles of the type (2.7), (2.8), such models were investigated in detail in [23–25] (mostly for $d = 2$, but beyond the scope of any perturbation theory).

4. $g_* = (\varepsilon - \xi/2)/3$, $w_* = \xi$; $\Omega_g = (\varepsilon - \xi/2)$, $\Omega_w = \xi$. This fixed point corresponds to a new nontrivial IR scaling regime, in which both the nonlinearities in the stochastic equation for φ are important; the corresponding critical dimensions depend essentially on both the RG expansion parameters ε and ξ and are calculated as double series in these parameters; see section 6. This behaviour reveals strong anisotropy and belongs to a new, completely non-equilibrium, universality class in the sense that the equal-time correlation functions are not given by a Gibbs measure $\exp\{-\mathcal{H}(\varphi)\}$ with a Hamiltonian of the type (2.3).

In figure 1, we show the regions of IR stability for all these fixed points in the ε–ξ plane, that is, the regions in which the eigenvalues $\Omega_{g,w}$ for a given fixed point are both positive.

In the leading-order approximation (5.3), all the boundaries of the regions of stability are given by straight lines; there are neither gaps nor overlaps between the different regions. However, experience with analogous two-parameter models (e.g. double expansion for the stochastic Navier–Stokes equation near two dimensions [42]) suggests that such a behaviour

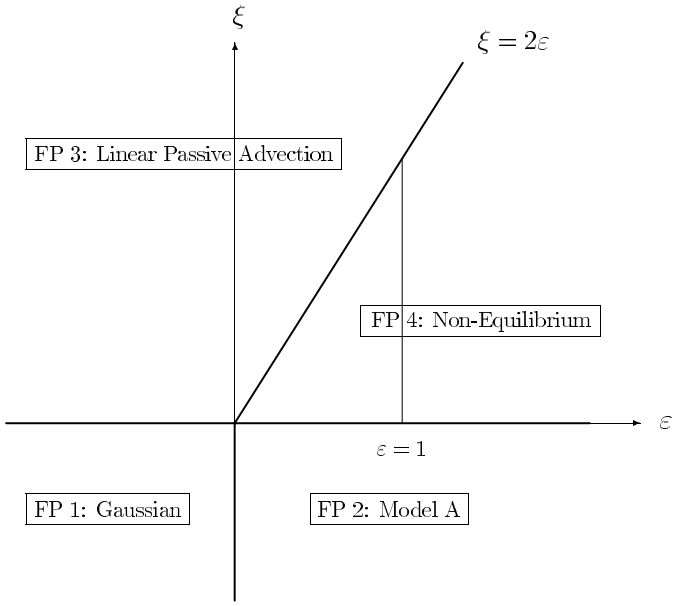

Figure 1. Regions of stability of the fixed points in the model (2.11).

can rather be an artefact of the leading-order approximation: the boundaries become curved and overlaps can appear if the higher order corrections in the β functions are taken into account. In our model, this definitely happens for the boundary between the regions of stability of the fixed points 2 and 4, as can be argued without practical calculation of the corrections to functions (5.3); see section 6.

One can see that the interval of the most realistic values of these parameters, $\varepsilon = 1$ ($d = 3$) and $0 < \xi < 2$ (see the remark above equation (2.9)), belongs completely to the region of stability of the most nontrivial fixed point 4. It is also worth noting that, for all fixed points, the coordinates g_*, w_* are positive in the regions of their IR stability, in agreement with the physical meaning of these parameters: w enters the amplitude in a pair correlation function and $g > 0$ is required for the stability of the static model (2.3).

6. Critical scaling and critical dimensions

Let F be some function of n independent arguments $\{x_1, \ldots, x_n\}$ satisfying the following scaling relation:

$$F(\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_n} x_n) = \lambda^{\alpha_F} F(x_1, \ldots, x_n) \tag{6.1}$$

with a set of constant coefficients (scaling dimensions) $\{\alpha_1, \ldots, \alpha_n, \alpha_F\}$ and any positive parameter $\lambda > 0$. Differentiating (6.1) with respect to λ and setting $\lambda = 1$ gives the first-order differential equation

$$\sum_{i=1}^{n} \alpha_i \mathcal{D}_i F(x_1, \ldots, x_n) = \alpha_F F(x_1, \ldots, x_n), \qquad \mathcal{D}_i = x_i \partial/\partial x_i, \tag{6.2}$$

whose general solution has the form

$$F(x_1, x_2, \ldots, x_n) = x_1^{\alpha_F/\alpha_1} \widetilde{F}\left(\frac{x_2}{x_1^{\alpha_2/\alpha_1}}, \ldots, \frac{x_n}{x_1^{\alpha_n/\alpha_1}}\right), \tag{6.3}$$

where $\widetilde{F}$ is an arbitrary function of $(n-1)$ arguments. Obviously, the dimensions are determined up to an overall constant factor (replace $\lambda \to \lambda^a$ in (6.1) or multiply (6.2) by a); this arbitrariness can be fixed e.g. by setting $\alpha_1 = 1$. If $\alpha_i = 0$ for some x_i, this variable is not dilated in (6.1) and the corresponding derivative in (6.2) is absent.

It is well known that the leading term of the large-scale asymptotic behaviour of a (renormalized) correlation function satisfies the RG equation (4.2) in which the renormalized couplings are replaced with their fixed-point values. In our case, this gives

$$\{\mathcal{D}_\mu - \gamma_u^* \mathcal{D}_u - \gamma_\sigma^* \mathcal{D}_\sigma - \gamma_\tau^* \mathcal{D}_\tau + N_\Phi \gamma_\Phi^*\} G_{N_\Phi} = 0, \tag{6.4}$$

where $\gamma_u^* = \gamma_u(g = g_*, w = w_*)$ and so on, and G_{N_Φ} is the renormalized correlation function with $N_\Phi = \{N_{\varphi'}, N_\varphi, N_v\}$ fields. The summation over all types of fields in the last term of (6.4) and analogous expressions below is implied; cf equation (3.1).

Canonical scale invariance of the function G_{N_Φ} with respect to the three independent canonical dimensions (see section 3) can be expressed by the differential equations of the form (for definiteness, we consider the momentum–frequency representation)

$$\{\mathcal{D}_\omega - \mathcal{D}_\sigma - N_{\varphi'} - N_v\} G_{N_\Phi} = 0, \tag{6.5}$$

$$\{\mathcal{D}_\perp + \mathcal{D}_\mu + 2\mathcal{D}_\sigma + 2\mathcal{D}_\tau + 2\mathcal{D}_u - (N_{\varphi'} + N_\varphi)(d-3)/2\} G_{N_\Phi} = 0, \tag{6.6}$$

$$\{\mathcal{D}_\parallel - 2\mathcal{D}_u - (N_{\varphi'} + N_\varphi)/2 + N_v\} G_{N_\Phi} = 0, \tag{6.7}$$

where $\mathcal{D}_\parallel = k_\parallel \partial/\partial k_\parallel$, $\mathcal{D}_\perp = k_\perp \partial/\partial k_\perp$ and the data from table 1 have been used.

Clearly, equation (6.4) corresponds to the scaling behaviour (6.1) of the function G_{N_Φ} upon the dilatation of the parameters σ, μ, u and τ and at fixed momentum and frequency variables; equation (6.5) deals with dilatation of the frequencies and other parameters at fixed momenta, and so on. We are interested in the critical scaling behaviour, that is, behaviour of the type (6.1) in which all the IR relevant parameters (momenta/coordinates, frequencies/times, deviation of the temperature from its critical value $\tau \propto (T - T_c)$) are dilated, while the IR irrelevant parameters (those which remain finite at the fixed point: σ, μ and u) are fixed [2, 3]. Thus, we combine equations (6.4)–(6.7) so that the derivatives with respect to the IR irrelevant parameters are eliminated; this gives the desired equation which describes the critical scaling behaviour:

$$\{\mathcal{D}_\perp + \Delta_\parallel \mathcal{D}_\parallel + \Delta_\omega \mathcal{D}_\omega + \Delta_\tau \mathcal{D}_\tau - N_\Phi \Delta_\Phi\} G_{N_\Phi} = 0. \tag{6.8}$$

Here $\Delta_\perp = 1$ is the normalization condition, while the critical dimensions of any other IR relevant parameter F are given by the general expression

$$\Delta_F = d_F^\perp + \Delta_\parallel d_F^\parallel + \Delta_\omega d_F^\omega + \gamma_F^* \tag{6.9}$$

with the canonical dimensions from table 1 and

$$\Delta_\omega = 2 + \gamma_\sigma^*, \qquad \Delta_\parallel = (2 + \gamma_u^*)/2. \tag{6.10}$$

Below we will concentrate on the two nontrivial fixed points 2 and 4; see section 5.

At the fixed point 2, where $w_* = 0$ exactly, all the anomalous dimensions γ_F^* in the RG equation (4.2), (4.3) coincide with their counterparts for the equilibrium model A. In particular, $\gamma_u^* = \gamma_4^* - \gamma_3^* = 0$ and therefore $\Delta_\parallel = 1$; the $SO(d)$ symmetry violated by the velocity ensemble is restored. Furthermore, from the fluctuation–dissipation theorem it follows that $\gamma_1^* = \gamma_2^*$; see e.g. [3, 5]. The standard notation for this equilibrium case is

$$\Delta_\varphi = d/2 - 1 + \eta/2, \qquad \Delta_\tau = 1/\nu, \qquad \Delta_\omega = z, \tag{6.11}$$

while for φ' the aforementioned relations for γ_F^* give $\Delta_{\varphi'} = \Delta_\varphi + z = d/2 - 1 + z + \eta/2$. The exponents η and ν can be found directly from the *static* model (2.3); they are well

known from the $4-\varepsilon, 2+\varepsilon$ and $1/N$ expansions, real-space RG (all augmented by various summations), high-temperature expansions for the Ising model (considered most reliable) and Monte Carlo simulations. The values recommended by [2, 3] are $\eta = 0.0375 \pm 0.0025$ and $\nu = 0.6310 \pm 0.0015$ (the Borel summation of fifth-order results). For z only two terms of the $4-\varepsilon$ expansion are known: $z = 2 + 0.726(1 - 0.1885\varepsilon)\eta$ [43]; there are also four-loop results in the real-space RG [44] and leading-order results in $2+\varepsilon$ and $1/N$ expansions; see the discussion in [3–5] and references therein.

Let us turn to the fixed point 4. We are going to find the critical dimensions entering the equations (6.8) and (6.10) to the second order of the generalized ε expansion, that is, the double expansion in ε and ξ with the convention that $\xi = O(\varepsilon)$. From relations (4.8) and (4.10) it follows that $\gamma_{\sigma,\varphi,\varphi'} = O(g^2)$ and $\gamma_u = O(w)$, so that in order to find $\gamma^*_{\sigma,\varphi,\varphi'}$ with the accuracy of $O(\varepsilon^2)$ it is sufficient to calculate the coordinate g_* only to the leading order $O(\varepsilon)$. At first sight, the second-order calculation of the coordinates g_*, w_* is needed to find the $O(\varepsilon^2)$ contribution in the dimension γ^*_u. However, this calculation can be avoided with the aid of the exact identity

$$\gamma^*_u = \xi + \gamma^*_\sigma, \tag{6.12}$$

which follows from the relations $\beta_w = w[-\xi - \gamma_w]$ in (4.5) and $\gamma_u + \gamma_w = \gamma_\sigma$ in (4.7) for any fixed point at which $\beta_w = 0$ and $w_* \neq 0$. Then from (6.10) it follows that

$$2\Delta_\parallel = \Delta_\omega + \xi \tag{6.13}$$

exactly, and from the explicit expressions (4.10) one obtains ($\bar\varepsilon \equiv \varepsilon - \xi/2$)

$$\begin{aligned}
\Delta_\varphi &= 1 - \bar\varepsilon/2 + (6b+1)\bar\varepsilon^2/486,\\
\Delta_{\varphi'} &= 3 - \bar\varepsilon/2 + (10b-1)\bar\varepsilon^2/72,\\
\Delta_\parallel &= 1 + \xi/2 + (6b-1)\bar\varepsilon^2/108,\\
\Delta_\omega &= 2 + (6b-1)\bar\varepsilon^2/54,\\
\Delta_\tau &= 2 - \bar\varepsilon/3
\end{aligned} \tag{6.14}$$

with corrections of order $O(\varepsilon^2)$ for Δ_τ and $O(\varepsilon^3)$ for the other dimensions; we recall that $b = \ln(4/3) \approx 0.287\ 683$. For the velocity field, relations (6.9), (6.10) and (6.12) can be combined to give the exact expression

$$\Delta_v = 1 + \gamma^*_\sigma - \gamma^*_u/2 = (\Delta_\omega - \xi)/2, \tag{6.15}$$

in agreement with the explicit factorized form of the velocity correlation function (2.7), (2.8).

It is worth noting that in the approximation (6.14) all the dimensions appear polynomial in the regulators ε and ξ (no contributions like e.g. ε^2/ξ or $\varepsilon^2/(\varepsilon - \xi/2)$) and depend on them only through the single combination $\bar\varepsilon \equiv \varepsilon - \xi/2$ (except for the simple second term in $\Delta_\parallel$). This reminds of the situation encountered earlier for a model of ferromagnets with extended quenched impurities, in which the two-loop critical dimensions appear analytic in the two UV regulators [37] although the RG functions and coordinates of the fixed points (calculated in the MS scheme) involve their ratios [36, 37]. Below we try to give some arguments that, at least in our model, this property will hold to all orders of the perturbation theory. In contrast, the dependence on the single parameter $\bar\varepsilon$ is a consequence of a relatively simple form of the RG functions in the approximation (4.10), and we do not expect it to hold in the higher orders.

As already mentioned at the end of section 3, the general theory of analytic regularization establishes that the anomalous dimensions can be made analytic in the full set of regulators, if a special renormalization scheme is chosen [41]. The coordinates of the fixed points, obtained

by solving equations (5.1), can nevertheless appear non-analytic already in the leading-order approximation, as it happens for the stirred Navier–Stokes equation near two dimensions [42, 45]. Then the critical dimensions, obtained by the substitution of those coordinates into the anomalous dimensions, can also become non-analytic. However, if the anomalous dimensions in any given-order approximation are analytic not only in the regulators, but also in the coupling constants, and if the coordinates of the fixed points happen to be analytic in the regulators in the *leading-order* approximation, then it can be directly verified that they can be systematically calculated as multiple series in the regulators *to all orders*, and, consequently, this will be equally true for the critical dimensions. This is exactly our case, as follows from the explicit leading-order approximations for g_* and w_* given in section 5.

If the MS scheme (as defined at the end of section 3) is used in the calculation, the non-analytic contributions in the regulators can occur in the renormalization constants and anomalous dimensions, but they must disappear from the critical dimensions, because the latter do not depend on the renormalization scheme. This probably explains the cancellations observed in [37] in the two-loop approximation.

The key difference with the problem studied in [42, 45] is that our model remains multiplicatively renormalizable for the special cases $g = 0, w \neq 0$ and $w = 0, g \neq 0$ (one interaction does not generate the other one as a counterterm in the renormalization procedure), and all the renormalization constants and anomalous dimensions are just double series in g and w. In contrast, the model studied in [45] is not closed with respect to renormalization if one of its coupling constants is set equal to zero. As a consequence, the coupling constant corresponding to this 'generated' counterterm enters the denominators of the anomalous dimensions, and the arguments given above do not work: the coordinates of the fixed points become non-analytic in the regulators already in the leading order [45].

It should be noted, however, that the general theory of renormalization and analytic regularization, which we took to apply to our case, was developed mostly in connection with local relativistic quantum field theories and their Euclidean counterparts. Its extension to field theoretic models, related to stochastic differential problems and involving quenched disorder, random velocity fields and so on, although supported by a great number of various examples, can hardly be rigorously justified in general. Thus, some doubts can remain about the validity of the above arguments to all higher orders of the perturbation theory. If non-analytic contributions nevertheless occur in the higher order corrections to expressions (6.14), the double expansion for the dimensions should be understood as the expansion in ε with $\xi = a\varepsilon, a = O(1)$ and possible singularities in its coefficients for some values of a.

It remains to note that the critical dimensions (6.14) coincide up to the order $O(\varepsilon)$ with their counterparts in (6.11) at the ray $\varepsilon > 0, \xi = 0$, the boundary between the regions of stability of the corresponding fixed points 2 and 4 (determined in section 5 in the first-order approximation for the β functions), but differ in order $O(\varepsilon^2)$. This is a clear indication that a straight boundary without gaps and overlaps is an artefact of the first-order approximation: the boundaries become curved and overlaps or gaps can appear when the higher order corrections to the β functions are taken into account, as happens in the analogous double expansion for the stochastic Navier–Stokes equation near two dimensions [42].

7. Discussion and conclusion

We have studied effects of turbulent mixing and stirring on the critical behaviour of a fluid system (binary mixture, nematic liquid crystal) with a purely relaxational dynamics of a non-conserved order parameter, known as model A [3–5]. The velocity was modelled by Gaussian statistics with vanishing correlation time and strongly anisotropic correlation

function $\propto\delta(t-t')\big/k_\perp^{d-1+\xi}$; see equations (2.7), (2.8). Such ensembles were employed earlier in [23–25] in the analysis of the two-dimensional passive turbulent advection (linear equation for the scalar field).

The model, originally described by a stochastic differential equation (2.1)–(2.4), can be reformulated as a multiplicatively renormalizable field theory (2.11), which allows one to apply the field theoretic RG to study its critical behaviour. The model reveals four different IR scaling regimes, related with the four different fixed points of the RG equations; their regions of stability in the ε–ξ plane are identified in the leading order. These regimes correspond to (1) Gaussian (free) model, (2) equilibrium critical dynamics (standard universality class of model A, interaction with the velocity field is irrelevant), (3) linear passive scalar advection (the φ^4 term in the Landau–Ginzburg Hamiltonian is irrelevant) and (4) the most nontrivial strongly anisotropic scaling regime in which both the interactions are important; it corresponds to a new non-equilibrium universality class.

It was shown that the equilibrium critical regime (model A) becomes unstable for the realistic range of parameters $d < 3$ and $0 < \xi < 2$, which includes the Kolmogorov spectrum ($\xi = 4/3$) and Batchelor limit ($\xi = 2$). It is replaced with the new non-equilibrium regime; the corresponding critical exponents are calculated to second order of the corresponding RG expansion, which in this case takes on the form of the double expansion in ε and ξ; explicit expressions are given in (6.14).

Let us discuss the consequences of the general scaling relations, derived in section 6, for the most interesting special case of the pair correlation function. They result in the scaling expression

$$\langle\varphi(\mathbf{x}+\mathbf{r},t+t')\varphi(\mathbf{x},t')\rangle = r_\perp^{-2\Delta_\varphi}\mathcal{F}\big(\tau_0 r^{\Delta_\tau}, t\big/r_\perp^{\Delta_\omega}, r_\parallel\big/r_\perp^{\Delta_\parallel}\big), \tag{7.1}$$

where $r_\perp = |\mathbf{r}_\perp|$, $r_\parallel = |\mathbf{r}_\parallel|$ and $\mathcal{F}$ is some scaling function. It is usually assumed that $\mathcal{F}$ has finite limits for $\tau_0 \propto (T - T_c) = 0$ (that is, exactly at the critical point) and/or for $t = 0$ (equal-time correlation function). Then from (7.1) one obtains

$$\langle\varphi(\mathbf{x}+\mathbf{r},t)\varphi(\mathbf{x},t)\rangle = r_\perp^{-2\Delta_\varphi}\widetilde{\mathcal{F}}\big(r_\parallel/r_\perp^{\Delta_\parallel}\big) \tag{7.2}$$

with another nontrivial function $\widetilde{\mathcal{F}}(x) = \mathcal{F}(0,0,x)$. The last two arguments in the scaling representation (7.1) can also be chosen in the form $r_\perp/L_\perp(t)$ and $r_\parallel/L_\parallel(t)$ with two different characteristic length scales

$$L_\perp(t) \sim t^{\alpha_\perp}, \qquad L_\parallel(t) \sim t^{\alpha_\parallel}, \qquad \alpha_\perp = 1/\Delta_\omega, \qquad \alpha_\parallel = \Delta_\parallel/\Delta_\omega, \tag{7.3}$$

with the exact relation $2\alpha_\parallel = 1 + \xi\alpha_\perp$ following from equation (6.13). For the most realistic values $\varepsilon = 1$ ($d = 3$) and $\xi = 4/3$ (Kolmogorov spectrum of the velocity) explicit results (6.14) give

$$\Delta_\omega \approx 2.0015, \qquad \alpha_\perp \approx 0.4996 \qquad \text{and} \qquad \alpha_\parallel \approx 0.833, \tag{7.4}$$

while for $\varepsilon = 1$ and $\xi = 2$ (Batchelor limit, smooth velocity field) in the same approximation one obtains

$$\Delta_\omega = 2, \qquad \alpha_\perp = 0.5 \qquad \text{and} \qquad \alpha_\parallel = 1, \tag{7.5}$$

with possible corrections from the $O(\varepsilon^3)$ terms in (6.14). It is worth noting that the $O(\varepsilon^2)$ contributions to these results are almost negligible, so that Δ_ω appears almost indistinguishable from its canonical value $\Delta_\omega = 2$. In contrast, the analogue of Fisher's exponent $\eta = \xi/2 + (6b+1)\bar{\varepsilon}^2/243$, determined from (6.14) using the 'equilibrium' relation (6.11), markedly deviates from its canonical (vanishing) value due to the $O(\xi)$ term: $\eta \approx 2/3$ for $\xi = 4/3$ and $\eta \approx 1$ for $\xi = 2$. This is reminiscent of the observation made in [9, 10]

(however, for a conserved order parameter and a non-random velocity) that the critical fluctuations are suppressed by the flow and the behaviour of the system becomes close to the mean-field limit in a strong shear; see also the discussion in [13].

The existence of two different length scales (7.3) with power-law dependence on the time was established in a number of studies within numerical simulations [15, 16], approximate analytical solutions [17] and exactly soluble simplified models [18]. As a rule, those authors dealt with binary mixtures in the coexistence (two-phase) region (finite and negative $\tau_0 \propto (T - T_c)$) in the presence of a uniform laminar shear flow, while our results refer to a system near its critical point and in a chaotic velocity ensemble. (For finite $\tau_0 < 0$, phase separation occurs at length scales comparable to the typical size of turbulent eddies, the situation which is much more difficult to achieve in practice in the vicinity of the critical point ($\tau_0 \simeq 0$), at least for binary mixtures; see also the discussion in [8].) Thus, *a priori* one should not have expected a good quantitative agreement for the exponents in (7.3). Surprisingly enough, our answers (7.4) and (7.5) for the exponents appear not inconsistent with the results $\alpha_\perp = 0.5$ and $\alpha_\parallel = 3/2$, derived earlier in [17] for a non-conserved order parameter in a uniform non-random shear within the so-called Ohta–Jasnow–Kawasaki approximation [46]. Although the values of $\alpha_\parallel$ are rather different in [17] and (7.4), (7.5), they are always markedly larger than $\alpha_\perp$. The same inequality $\alpha_\parallel > \alpha_\perp$ for the exponents was also established in two dimensions [17] and for exactly soluble models [18].

Let us briefly discuss the general case (2.5) with $\mathbf{u} \neq 0$. Nonvanishing mean velocity $\mathbf{u}$ gives rise to the additional term $\sigma_0\varphi'(u_i\partial_i)\varphi = \sigma_0\varphi'(u\partial_\parallel)\varphi$ with $u = |\mathbf{u}|$ in the action (2.11), which simply results in the replacement $\omega \to \omega - \mathbf{u}\cdot\mathbf{k} = \omega - uk_\parallel$ in the propagators (2.15), (2.16) and $r_\parallel \to r_\parallel + ut$ in the final scaling expressions like (7.1). This fact can be compared with the observation made in [16, 17] that, for $\mathbf{u} \neq 0$, the proper scaling variables are not simply related to parallel and perpendicular directions. The dependence on $\mathbf{u}$ disappears at $t = 0$, that is, in the equal-time correlation function (7.2).

Another interesting quantity is the 'crossover exponent' χ in the relation $\delta T_c \propto Re^\chi$ between the Reynolds number Re and the shift δT_c of the critical temperature due to the mixing, with experimental estimates $\chi \sim 1.4$–2.1 [12]. In the RG framework, this exponent can be identified [7, 8] as $\chi = \nu|\Omega_{\min}|$, where $\nu \simeq 0.63$ is the classical critical exponent (6.11) for the Landau–Ginzburg model (2.3) and $\Omega_{\min}$ is the minimal (maximal by the modulus) negative eigenvalue of the Ω matrix (5.2) at the equilibrium scaling regime (model A or, in our notation, fixed point 2). In our case $\Omega_{\min} = \Omega_w = -\xi$ for point 2; see section 5. This gives $\chi \simeq 1.2$ for the Kolmogorov spectrum ($\xi = 4/3$) and $\chi \simeq 1.26$ for the Batchelor limit, which is better than the estimate $\chi \simeq 0.8$ obtained in [10] for strongly anisotropic non-random shear but worse than the RG result $\chi \simeq 1.74$ obtained in [7, 8] for a random isotropic velocity ensemble with the velocity spectrum $\propto 1/k^2$. Of course, the disagreement can be explained by the non-conservation of the order parameter in our model. (In this connection it should be mentioned that the discussion of section VI in [8] for the general exponent in the velocity correlation function, denoted as $1/k^{2+a\varepsilon}$ in equation (6.1) of [8], contains an error: the second β function in (6.8) must be $\beta_\lambda = -\widetilde{\lambda}_R\{(1+a)\varepsilon - \cdots\}$. Thus, the conclusions made in the following discussion about the independence of the critical exponents on a (in our notation, on the relation between the two RG expansion parameters ε and ξ) can be erroneous and must be revisited.)

It remains to note that for $d < 3$ and not too small ξ, the fixed point 4 becomes unstable while point 3 becomes IR attractive (see figure 1), the φ^4 interaction in (2.3) becomes irrelevant and the IR behaviour of the model coincides with that of the linear passive scalar advected by the anisotropic Gaussian velocity ensemble (2.7), (2.8). For $d = 2$, this regime was investigated in detail in [23–25].

We may conclude that our simplified model of a non-conserved order parameter and a Gaussian velocity ensemble captures important characteristics of a real second-order phase transition in a stirred fluid system: persistence of a critical scaling regime; emergence of a new non-equilibrium universality class with a new set of critical exponents, rather different from the classical ones; existence (for a strongly anisotropic velocity ensemble) of two different length scales (with a power-law time dependence), and so on. Further investigation should take into account conservation of the order parameter and its interaction with other thermodynamical degrees of freedom (mode–mode coupling), compressibility, non-Gaussian character and finite correlation time of the velocity field, and so on. This work is now in progress.

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Appendix A. Consequences of the Galilean symmetry

In this appendix, we will explore consequences of the Galilean symmetry for the renormalization of the model (2.11). The models with synthetic Gaussian velocity ensembles are, as a rule, not invariant with respect to the Galilean transformations. Nevertheless, if the velocity is not correlated in time, nontrivial parts of the 1-irreducible correlation functions appear invariant (more precisely, see below), and the Galilean symmetry can be used to restrict the form of the counterterms. In this sense, the symmetry of the counterterms is higher than the symmetry of the action functional.

For most of the following discussion, a precise form of the nonlinearity in (2.1) is unessential; it is only important that it is consistent with Galilean symmetry. The velocity field will be taken divergence-free, Gaussian, with zero mean and the correlator

$$\langle v_i(t,\mathbf{x})v_j(t',\mathbf{x}')\rangle = \delta(t-t')D_{ij}(\mathbf{x}-\mathbf{x}'). \tag{A.1}$$

Our model (2.7), (2.8) corresponds to a special choice of the function D_{ij}, but in the following its precise form is also unessential.

Field theoretic formulation (2.11) means that the generating functionals of total ($G(A)$) and connected ($W(A)$) correlation functions of the original stochastic problem can be represented by the functional integral of the form

$$G(A) = \exp W(A) = \int \mathcal{D}\Phi \exp\{\mathcal{S}(\Phi) + A\Phi\}. \tag{A.2}$$

Here and below, we denote by $\Phi = \{\varphi', \varphi, \mathbf{v}\}$ the full set of fields and by $A = \{A_{\varphi'}, A_{\varphi}, \mathbf{A}_v\}$ the full set of sources; in the expressions like

$$A\Phi = \sum_{\Phi} \int \mathrm{d}x\, A(x)\Phi(x)$$

summation over all types of the fields, integration over their arguments $x = \{t, \mathbf{x}\}$ and summation over their vector indices are always understood. All the normalization factors are included into the functional differential $\mathcal{D}\Phi = \mathcal{D}\varphi'\mathcal{D}\varphi\mathcal{D}\mathbf{v}$; the normalization $G(0) = 1$ is implied.

The Galilean transformation is defined as

$$v_i(t,\mathbf{x}) \to \widetilde{v}_i(t,\mathbf{x}) = v_i(t,\mathbf{x}+\mathbf{u}t) - u_i$$

for the velocity and

$$\Phi(t,\mathbf{x}) \to \widetilde{\Phi}(t,\mathbf{x}) = \Phi(t,\mathbf{x}+\mathbf{u}t) \tag{A.3}$$

for the other fields; here $\mathbf{u}$, the parameter of the transformation, is an arbitrary constant vector. For the strongly anisotropic ensemble (2.7), (2.8), the vector $\mathbf{u}$ must be parallel to $\mathbf{v} \sim \mathbf{n}$; then the shift of the arguments in (A.3) reduces to $x_\parallel \to x_\parallel + u$.

The part of the action (2.11) which corresponds to the stochastic problem (2.1) at fixed $\mathbf{v}$ is clearly invariant: one has to substitute $\Phi \to \widetilde{\Phi}$ and make the change of variables $\mathbf{x} + \mathbf{u}t \to \mathbf{x}$; the additional terms $\varphi'(u_i\partial_i)\varphi$ coming from the contribution with ∂_t and from the nonlinearity cancel each other in the covariant combination $\varphi'\nabla_t\varphi$. If the velocity were governed by the stochastic Navier–Stokes equation with a time-decorrelated random force, the total action would also be invariant (see e.g. [3]), but for our synthetic ensemble (A.1) variation of the action $\mathcal{S}_v(\mathbf{v})$ in (2.13) is nontrivial:

$$\mathcal{S}_v(\widetilde{\mathbf{v}}) = \mathcal{S}_v(\mathbf{v}) + uD^{-1}v + O(u^2). \tag{A.4}$$

In the detailed notation,

$$uD^{-1}v = \int \mathrm{d}t \int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{x}' u_i D^{-1}_{ij}(\mathbf{x}-\mathbf{x}')v_j(t,\mathbf{x}'),$$

where D^{-1} is the inverse linear operation for D in (A.1) on the transverse subspace.

We stress that for the validity of (A.4) it is crucial that the correlator (A.1) involves the δ function in time. Indeed, substitution $\mathbf{v} \to \widetilde{\mathbf{v}}$ in $\mathcal{S}_v$ produces the term

$$\int \mathrm{d}t \int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{x}' v_i(t,\mathbf{x}+\mathbf{u}t) D^{-1}_{ij}(\mathbf{x}-\mathbf{x}')v_j(t,\mathbf{x}'+\mathbf{u}t), \tag{A.5}$$

which gives $\mathcal{S}_v(\mathbf{v})$ after the change of variables $\mathbf{x}+\mathbf{u}t \to \mathbf{x}$, $\mathbf{x}'+\mathbf{u}t \to \mathbf{x}'$ due to the fact that both the fields in (A.5) have the same time argument and therefore the argument of D^{-1} remains unchanged. For a finite correlation time, expression (A.5) would involve the double time integral, the argument of D^{-1} would be shifted by $\mathbf{u}(t-t')$ and the original action $\mathcal{S}_v$ on the right-hand side of (A.4) would not be formed.

Let us make the substitution $\Phi \to \widetilde{\Phi}$ in the functional integral (A.2). This is just a change of integration variables, its Jacobian equals unity, so the integral

$$G(A) = \int \mathcal{D}\Phi \exp\{\mathcal{S}(\widetilde{\Phi}) + A\widetilde{\Phi}\} \tag{A.6}$$

is in fact independent of the parameter $\mathbf{u}$ from (A.3). In particular, this means that its first variation with respect to $\mathbf{u}$ vanishes. Let us denote by $\delta_u F(\Phi)$ the linear-in-$\mathbf{u}$ term in the Galilean transformed quantity $F(\widetilde{\Phi})$. Then we have $\delta_u\mathbf{v} = (u\partial)\mathbf{v} - \mathbf{u}$ for the velocity, $\delta_u\Phi = (u\partial)\Phi$ for the other fields and $\delta_u\mathcal{S}(\Phi) = \delta_u\mathcal{S}_v(\mathbf{v}) = uD^{-1}v$ for the action functional. Substituting these expressions into the first variation of (A.6) gives the identity

$$\int \mathcal{D}\Phi\{A(u\partial)\Phi - uA_v + uD^{-1}v\}\exp\{\mathcal{S}(\Phi) + A\Phi\} = 0 \tag{A.7}$$

with implied summations over all types of fields, integrations and so on. The fields can be taken outside the integral in (A.7) as variational derivatives with respect to the corresponding sources, $\Phi \to \delta/\delta A$, which gives for the functional $W(A)$ from (A.2) the following differential equation in variational derivatives:

$$A(u\partial)\frac{\delta W(A)}{\delta A} - uA_v + uD^{-1}\frac{\delta W(A)}{\delta A_v} = 0. \tag{A.8}$$

It is well known that the generating functional $\Gamma(\Phi)$ of the 1-irreducible correlation functions (sometimes referred to as 'effective action') is obtained from $W(A)$ as the functional Legendre transform with respect to the sources A (see e.g. [2, 3]):

$$\Gamma(\Phi) = W(A) - A\Phi, \qquad \frac{\delta W(A)}{\delta A} = \Phi, \qquad \frac{\delta \Gamma(\Phi)}{\delta \Phi} = -A. \tag{A.9}$$

Here the sources are (implicitly) expressed as functions of the fields for a given $W(A)$ using the second relation, while the third relation explicitly determines A in terms of Φ for a given $\Gamma(\Phi)$. Substituting (A.9) into (A.8) gives the following equation for $\Gamma(\Phi)$:

$$\frac{\delta \Gamma(\Phi)}{\delta \Phi}(u\partial)\Phi - u\frac{\delta \Gamma(\Phi)}{\delta v} = uD^{-1}v. \tag{A.10}$$

The left-hand side obviously represents the first variation $\delta_u\Gamma(\Phi)$ of the functional (A.9) with respect to the Galilean transformation (A.3) of its functional arguments, while the right-hand side is nothing but the variation $\delta_u\mathcal{S}(\Phi)$ of the action (2.11). It is well known that the functional (A.9) can be represented as the sum $\Gamma(\Phi) = \mathcal{S}(\Phi) + \bar{\Gamma}(\Phi)$ of the action $\mathcal{S}(\Phi)$, which contains all the tree ('loopless') graphs and the terms not represented by graphs, and the nontrivial part $\bar{\Gamma}(\Phi)$ which contains all the graphs with loops (and hence those with all possible UV divergences); see e.g. [2, 3]. From (A.10) we conclude that the total non-invariance of $\Gamma(\Phi)$ is brought about by the action term, while the second contribution appears invariant: $\delta_u\bar{\Gamma}(\Phi) = 0$.

The last relation holds for arbitrary values of the model parameters, including d, ξ and the coupling constants (2.18). Therefore, it remains valid in the perturbation theory and is preserved by the renormalization procedure. We thus may conclude that the contribution of the counterterms (determined by the nontrivial term $\bar{\Gamma}(\Phi)$) must also be Galilean invariant, in spite of the fact that the total functional $\Gamma(\Phi)$ is not. This justifies the statements made in the analysis of the renormalization of our model in section 3: the counterterm $\varphi'\varphi v^2$, allowed by the dimension, is not invariant and therefore it is forbidden; the counterterms $\varphi'\partial_t\varphi$ and $\varphi'(v_i\partial_i)\varphi$ can appear only in the form of the Galilean covariant combination $\varphi'\nabla_t\varphi$.

Appendix B. Calculation of the Feynman diagrams

In this appendix, we will briefly discuss the main points concerning the calculation of the renormalization constants and the corresponding Feynman diagrams. In order to find the anomalous dimensions (4.6) in the approximation (4.9), (4.10), one has to calculate the 1-irreducible correlation functions $\langle\varphi'\varphi'\rangle$, $\langle\varphi'\varphi\rangle$ to the two-loop order and $\langle\varphi'\varphi\varphi\varphi\rangle$ to the one-loop order of the renormalized perturbation theory. The corresponding diagrammatic expressions are given in figures 2–4 (we do not show some of the diagrams, which are topologically possible but vanish for special reasons; see below).

The wavy lines denote the pair correlator of the velocity (2.8). The bare propagators (2.15), (2.16) are denoted by solid lines: the line without a slash denotes the propagator $\langle\varphi\varphi\rangle_0$ while the line with one slash denotes $\langle\varphi\varphi'\rangle_0$; the slashed ends correspond to the response field φ'. The propagator $\langle\varphi'\varphi'\rangle_0$ vanishes in any dynamic model of the type (2.11), so the solid line with two slashed ends does not occur in the diagrams. There are two types of vertices (shown by thick dots): the quartic one corresponds to the interaction $\varphi'\varphi^3$ and the triple one corresponds to $\varphi'(v\partial_\parallel)\varphi$. The symmetry coefficients are shown for the one-component scalar field ($N = 1$).

All the diagrammatic elements should be expressed in renormalized variables using the relations (3.3)–(3.7). However, in our approximation Z's should be retained (with the appropriate accuracy in g and w) only in the bare terms of the functions $\langle\Phi\cdots\Phi\rangle$, while in the

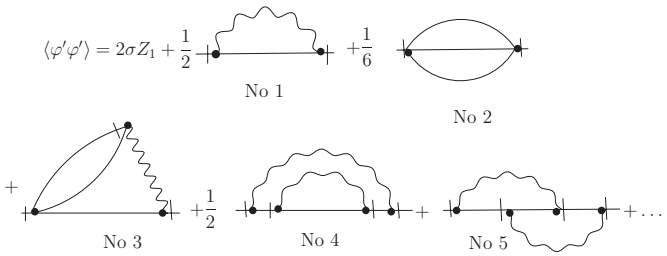

Figure 2. 1-irreducible function $\langle\varphi'\varphi'\rangle$: relevant one- and two-loop diagrams.

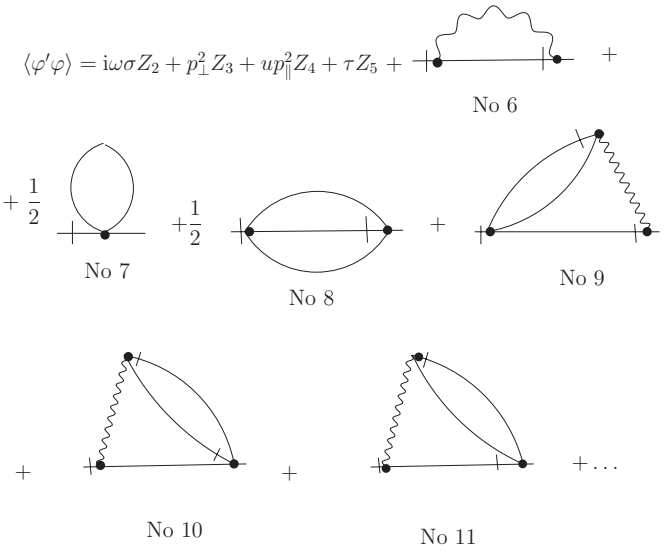

Figure 3. 1-irreducible function $\langle\varphi'\varphi\rangle$: relevant one- and two-loop diagrams.

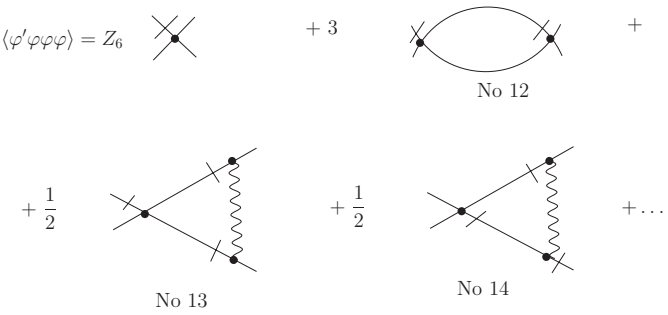

Figure 4. 1-irreducible function $\langle\varphi'\varphi\varphi\varphi\rangle$: one-loop approximation.

diagrams they should be replaced with unities. In other words, the passage to renormalized variables in the diagrams is provided by the simple substitutions $\sigma_0 \to \sigma, u_0 \to u, \tau_0 \to \tau, g_0 \to g\mu^\varepsilon$ and $w_0 \to w\mu^\xi$.

In the practical calculation we use the MS scheme, where the renormalization constants are independent of the specific choice of the IR regularization. It is then possible to calculate the constants directly in the critical ('massless') model, that is, at $\tau = 0$ in the renormalized analogues of the lines (2.15), (2.16). Then, in the calculation of the constants Z_i with $i \neq 5$, the diagrams involving self-contracted propagators $\langle\varphi\varphi\rangle_0$ can be treated as zero and, for this reason, are not shown in the figures. The only exception is made for the one-loop diagram 7 with a self-contracted solid line in the function $\langle\varphi\varphi'\rangle$ needed for the calculation of Z_5, in which τ should be retained. Depending of the type of a diagram and specific way of calculation, the IR regularization is either provided by the sharp cut-off or not needed at all (see below). The diagrams with self-contracted lines $\langle\varphi\varphi'\rangle_0$ should also be discarded according to the general rules of the diagrammatic technique for the dynamic models of the type (2.11); see [2, 3].

Owing to specific properties of our model, many diagrams still shown in the figures vanish identically or appear UV finite and therefore do not contribute to the renormalization constants.

Since the velocity field is transverse (divergence-free), the derivative $\partial_\parallel$ at the triple vertex can, if desired, be moved onto the response field using the integration by parts; see equation (2.12). Thus, in any diagram involving N *external* vertices of the type $\varphi'(v\partial_\parallel)\varphi$, the factor $p_\parallel^N$ with external momenta $p_\parallel$ will be taken outside the corresponding integrals over the internal momenta and frequencies. This reduces the dimension of the integrand and can make the diagram UV finite. In particular, this makes UV finite diagrams 1, 3, 4, 13 and 14 (the latter in fact vanishes, see below).

Diagrams 5, 10, 11 and 14 effectively involve closed circuits of retarded propagators $\langle\varphi\varphi'\rangle_0$ (self-contracted chains of step functions; see equation (2.15)) and therefore also vanish. Such an effect is well known for dynamic models of the type (2.11) and is related to causality; see [2, 3]. However, for diagrams 11 and 14 it is crucial here that the propagator of the velocity involves the δ function in time. These arguments, however, do not apply to the one-loop diagram 6, which requires more accurate consideration (see below).

There are more diagrams in the functions $\langle\varphi'\varphi'\rangle$ and $\langle\varphi'\varphi\rangle$ having the same topology as diagrams 4 and 5, with another placements of the slashes. They also vanish because of the two reasons explained above, and we do not show them in the figures.

As a result, in the functions $\langle\varphi'\varphi'\rangle$ and $\langle\varphi'\varphi\varphi\varphi\rangle$ only diagrams 2 and 12 are divergent.

Integration over times (or frequencies) in diagram 9 leads to the expression (up to a numerical factor and with implied IR cut-offs)

$$p_\parallel \int \mathrm{d}\mathbf{k} \int \mathrm{d}\mathbf{q} \frac{\delta(k_\parallel) q_\parallel}{[(\mathbf{p}+\mathbf{k})^2 + q^2 + (\mathbf{q}+\mathbf{k})^2] q^2 k_\perp^{d-1+\xi}}, \tag{B.1}$$

where one external momentum $p_\parallel$ has already appeared as an overall factor. Due to the presence of the δ function in the integrand, one can replace $(\mathbf{p}+\mathbf{k})^2 \to p_\parallel^2 + (\mathbf{p}_\perp + \mathbf{k}_\perp)^2$ in the denominator. Thus, the expansion of the integral in small momenta begins only with quadratic terms: $p_\perp^2$ (due to the persisting $SO(d-1)$ invariance) and $p_\parallel^2$. The total expression (B.1) can only begin with cubic terms and is therefore UV finite, because the possible superficial divergence of the function $\langle\varphi\varphi'\rangle$ must be quadratic; see section 3. It remains to note that in the isotropic case the UV finiteness of such a diagram would be guaranteed by the transversality of the velocity propagator; see the remark in [8].

Thus, we are left with the five UV divergent diagrams 2, 6, 7, 8 and 12.

The analytic expression for diagram 6 has the form

$$D_6(p) = (\mathrm{i}\sigma p_\parallel)^2 \int \frac{\mathrm{d}\omega}{(2\pi)} \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^d} D_v(k) \frac{1}{-\mathrm{i}\sigma\omega + \epsilon(\mathbf{p}-\mathbf{k})} \tag{B.2}$$

with D_v from (2.8), the prefactor coming from the vertices and $\epsilon(\mathbf{k}) = \mathbf{k}_\perp^2 + u\mathbf{k}_\parallel^2$ coming from (2.15); the result is independent of the external frequency. Integration over ω involves the indeterminacy

$$\int \frac{\mathrm{d}\omega}{(2\pi)} \frac{1}{-\mathrm{i}\sigma\omega + \epsilon(\mathbf{p}-\mathbf{k})} = \sigma^{-1}\theta(0), \tag{B.3}$$

where $\theta(0)$ is the step function at the origin. This indeterminacy reflects the details of the velocity statistics lost in the white-noise limit (2.7) and it should be carefully resolved; see e.g. the discussion in the appendix of [21]. In our case, the function $\delta(t-t')$ should be understood as the limit of a narrow function which is necessarily symmetric in $t \leftrightarrow t'$, because (2.7) is a pair correlation function. Thus, the quantity in (B.3) must be unambiguously defined by half the sum of the limits: $\theta(0) = 1/2$. Then after the trivial integration over $\mathbf{k}_\parallel$ and using (3.4) one obtains

$$D_6(p) = -p_\parallel^2 \frac{wu\mu^\xi}{2} \int \frac{\mathrm{d}\mathbf{k}_\perp}{(2\pi)^{(d-1)}} \frac{1}{k_\perp^{(d-1+\xi)}}. \tag{B.4}$$

Finally, the integration over $\mathbf{k}_\perp$ gives

$$D_6(p) = -p_\parallel^2 wu(\mu/m)^\xi \frac{S_{d-1}}{2(2\pi)^{(d-1)}} \frac{1}{\xi} = -p_\parallel^2 \frac{wu}{4\pi^2} \frac{1}{\xi} + \text{UV finite part}, \tag{B.5}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ with Euler's Γ function is the surface area of the unit sphere in the d-dimensional space.

As already discussed at the end of section 3, in our model there are subtleties in defining the MS scheme. In our calculations we always assumed that $\varepsilon \sim \xi$, treated the combinations like ε/ξ as UV finite and did not include them into the renormalization constants; this is implied in the last equality in (B.5). However, in the leading-order approximations these subtleties are not too important. In particular, another (and eventually equivalent) possibility is to include the factor $S_{d-1}/2(2\pi)^{(d-1)}$ (and not only its value at $\varepsilon = 0$) into the definition of the new coupling constant $\tilde{w}$; see the text below equation (4.9). This will include all powers of the ratio ε/ξ from (B.5) into the corresponding renormalization constant Z_4 without changing the anomalous dimension γ_4.

The remaining diagrams 2, 7, 8 and 12 do not involve the velocity propagator $\langle \mathbf{vv} \rangle_0$ and can be reduced to the well-known diagrams of the isotropic model A. Consider diagram 8 as an example. The corresponding analytic expression can be represented in the form

$$\lambda^2 I\big(\omega, p_\perp^2 + up_\parallel^2, u\big) = \lambda^2 u^{-1} I\big(\omega, p_\perp^2 + up_\parallel^2, u = 1\big). \tag{B.6}$$

Here, ω and $\mathbf{p} = \mathbf{p}_\perp + \mathbf{p}_\parallel$ are the external frequency and momentum, λ^2 comes from the vertex factors and $I(\cdots)$, after the integrations over the internal times or frequencies, is represented as a double integral over the two integration momenta, say, $\mathbf{k}$ and $\mathbf{q}$. All the momenta enter the integrand via the functions of the type $\epsilon(\mathbf{k}) = \mathbf{k}_\perp^2 + u\mathbf{k}_\parallel^2$ coming from expressions (2.15), (2.16) with the replacements $u_0 \to u$ and $\tau_0 \to 0$ (see the discussion in the beginning of the appendix). This explains the fact that $I(\cdots)$ depends on $\mathbf{p}$ through the only scalar argument $p_\perp^2 + up_\parallel^2$. This also gives the second equality in (B.6) after the rescaling of the integration momenta $u\mathbf{k}_\parallel \to \mathbf{k}_\parallel, u\mathbf{q}_\parallel \to \mathbf{q}_\parallel$. The factor u^{-1} arises from the Jacobians and combines with λ^2 to give the prefactor $\lambda^2 u^{-1} = g^2$; see equation (2.17). However, the quantity on the right-hand side of (B.6) is nothing other than the analytic expression for diagram 8 in the isotropic ($u = 1$) model A, while the dependence on u persists in its modified momentum argument $p^2 \to p_\perp^2 + up_\parallel^2$.

The isotropic integral is represented in the form

$$I(\omega, p, u = 1) = \{-\mathrm{i}\sigma\omega\mathcal{A} + p^2\mathcal{B}\}\frac{1}{\varepsilon} + \text{UV finite part}$$

with known (see e.g. [43]) dimensionless coefficients $\mathcal{A}$, $\mathcal{B}$, which for (B.6) gives

$$\lambda^2 I\big(\omega,\, p_\perp^2 + up_\parallel^2,\, u\big) = g^2\big\{-\mathrm{i}\sigma\omega\mathcal{A} + \big(p_\perp^2 + up_\parallel^{\,2}\big)\mathcal{B}\big\}\frac{1}{\varepsilon} + \text{UV finite part}. \tag{B.7}$$

This expression fully determines contribution of diagram 8 to the renormalization constants $Z_{2,3,4}$ in our model (3.3). Similar considerations determine the contributions of diagrams 2, 7 and 12 to the constants $Z_{1,5,6}$ in terms of the known coefficients for model A. The latter are well known, but for completeness we will discuss the corresponding calculational techniques which proved to be useful in the three-loop calculation in model A [43] and might be interesting in itself.

Two key points are as follows: the convolution of two functions of the form

$$F(\alpha; a) \equiv (-\mathrm{i}\omega a + k^2)^{-\alpha} \tag{B.8}$$

is a function of the same form,

$$F(\alpha; a) * F(\beta; b) = \theta(ab) K_2(\alpha, \beta; a, b) F(\alpha + \beta - d/2 - 1; a + b) \tag{B.9}$$

with the coefficient

$$K_2(\alpha, \beta; a, b) = a^{d/2-\alpha} b^{d/2-\beta} (a + b)^{\alpha+\beta-d-1} \frac{\Gamma(\alpha + \beta - d/2 - 1)}{(4\pi)^{d/2}\Gamma(\alpha)\Gamma(\beta)},$$

while the product of two such functions can be represented as a single integral of a function of the same form with the aid of the generalized Feynman formula:

$$F(\alpha; a) \cdot F(\beta; b) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \mathrm{d}s\, s^{\alpha-1}(1 - s)^{\beta-1} F(\alpha + \beta; as + b(1 - s)). \tag{B.10}$$

Equation (B.9) can be obtained from the fact that in the $\{t, \mathbf{x}\}$ representation function (B.8) takes on the form

$$F(\alpha; a) \to \frac{\theta(t\,\mathrm{sign}(a)) a^{d/2-\alpha}}{(4\pi)^{d/2}\Gamma(\alpha)} t^{\alpha-d/2-1} \exp\left\{-\frac{ax^2}{4t}\right\}, \tag{B.11}$$

and the product of such functions (which corresponds to the convolution of their Fourier transforms) is obviously a function of the same form. Note that for $ab < 0$ the convolution (B.9) vanishes because it corresponds to the product of a retarded and an advanced function of the form (B.11). In the same manner, for the convolution of three functions (B.8) one obtains

$$F(\alpha; a) * F(\beta; b) * F(\gamma; c) = K_3(\alpha, \beta, \gamma; a, b, c) F(\alpha + \beta + \gamma - d - 2) \tag{B.12}$$

with the coefficient

$$K_3(\alpha, \beta, \gamma; a, b, c) = \theta(ab)\theta(bc) \frac{\Gamma(\alpha + \beta + \gamma - d - 2)}{(4\pi)^{d/2}\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \times a^{d/2-\alpha} b^{d/2-\beta} c^{d/2-\gamma} (a + b + c)^{\alpha+\beta+\gamma-3d/2-2},$$

and so on.

In the notation (B.8), the bare propagators of the scalar fields φ, φ' can be written as

$$\langle\varphi\varphi'\rangle_0 = F(1, \sigma), \qquad \langle\varphi'\varphi\rangle_0 = F(1, -\sigma), \qquad \langle\varphi\varphi\rangle_0 = \int_{-\sigma}^{\sigma} \mathrm{d}u\, F(2; u).$$

In the last expression, we used equation (B.10) with $a = -b = \sigma$, $\alpha = \beta = 1$ and introduced the new integration variable as $u = \sigma(1 - 2s)$.

Consider again diagram 8 as an example of the calculation of the pole part in ε. It is convenient to set $\sigma = 1$ (the dependence on σ can easily be restored by dimensionality) and discard the factors like $(4\pi)^{-d/2}$ which always combine with g to form the new coupling

constant $\tilde{g}$; see the text below equation (4.9). Then the analytic expression for diagram 8 reads (we omit the factor g^2)

$$D_8 = \int_{-1}^{1} \mathrm{d}u_1 \int_{-1}^{1} \mathrm{d}u_2\, F(2,u_1) * F(2,u_2) * F(1,1). \tag{B.13}$$

Applying the reference formula (B.12) to the integrand of (B.13) gives

$$D_8 = \Gamma(3-d) \int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2\, u_1^{2-d/2} u_2^{2-d/2} (u_1+u_2+1)^{3-3d/2} F(3-d, u_1+u_2+1). \tag{B.14}$$

Note that only positive values of $u_{1,2}$ survive in (B.14) due to the θ functions in (B.12).

In order to extract the pole part of (B.14), it is sufficient to replace $\Gamma(3-d) = \Gamma(-1+\varepsilon) \to -1/\varepsilon$ and to set $\varepsilon = 0$ in the integrand; this gives

$$D_8 = -\frac{1}{2\varepsilon}[-\mathrm{i}\omega I_2 + p^2 I_3] + \text{UV finite part}, \tag{B.15}$$

with the coefficients

$$I_n = \int_0^1 \mathrm{d}u_1 \int_0^1 \mathrm{d}u_2 \frac{1}{(u_1+u_2+1)^n} = \int_0^2 \mathrm{d}u\, p(u) \frac{1}{(1+u)^n} \tag{B.16}$$

and the weight function

$$p(u) = \begin{cases} u, & \text{for } u < 1, \\ 2-u, & \text{for } u > 1. \end{cases}$$

The integrals in (B.16) are easily calculated directly, but it is instructive to use the following trick, which proved extremely useful in the higher order calculations. Let us introduce the generating function

$$I(z) = \int_0^2 \mathrm{d}u\, p(u) \frac{1}{z+u}. \tag{B.17}$$

It is easily seen that the integrals I_2 and I_3 from (B.16) are the second and third coefficients in the expansion of the quantity $I(1-\delta)$ in δ. Calculation of the integral (B.17) up to irrelevant constant terms gives

$$I(z) \simeq z \ln \frac{z(z+2)}{(z+1)^2} + 2 \ln \frac{z+2}{z+1},$$

and the expansion

$$I(1-\delta) = \text{const} + \delta \ln(4/3) + \delta^2/6 + O(\delta^3)$$

gives the desired coefficients $I_{2,3}$ in (B.15), which in their turn determine the coefficients $\mathcal{A}$, $\mathcal{B}$ in (B.7).

Once a renormalization constant Z_i has been calculated, the corresponding anomalous dimension is readily found from the relation

$$\gamma_i = (\beta_g \partial_g + \beta_w \partial_w) \ln Z_i = -(\varepsilon \mathcal{D}_g + \xi \mathcal{D}_w) \ln Z_i. \tag{B.18}$$

In the first equality, we used definition (4.4), expression (4.3) for the operation $\widetilde{\mathcal{D}}_\mu$ in renormalized variables and the fact that Z's depend only on the two completely dimensionless coupling constants g and w. In the second equality, we retained only the leading-order terms in the β functions (4.5), which is sufficient for our approximation. The factors ε and ξ in (B.18) cancel the corresponding poles contained in $\ln Z_i$, which leads to the final UV finite expressions for the anomalous dimensions, given in (4.9) and (4.10).

References

[1] Fisher M E 1965 *The Nature of Critical Points* (Boulder, CO: University of Colorado Press)
Stanley H 1971 *Introduction to Phase Transitions and Critical Phenomena* (Oxford: Clarendon)
[2] Zinn-Justin J 1989 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)
[3] Vasil'ev A N 2004 *The Field Theoretic Renormalization Group in Critical Behavior Theory and Stochastic Dynamics* (Boca Raton, FL: Chapman and Hall/CRC Press)
[4] Halperin B I and Hohenberg P C 1977 *Rev. Mod. Phys.* **49** 435
[5] Folk R and Moser G 2006 *J. Phys. A: Math. Gen.* **39** R207
[6] Khmel'nitskii D E 1975 *Sov. Phys.—JETP* **41** 981
Shalaev B N 1977 *Sov. Phys.—JETP* **26** 1204
Jayaprakash C and Katz H J 1977 *Phys. Rev.* B **16** 3987
Janssen H-K, Oerding K and Sengespeick E 1995 *J. Phys. A: Math. Gen.* **28** 6073
[7] Satten G and Ronis D 1985 *Phys. Rev. Lett.* **55** 91
[8] Satten G and Ronis D 1986 *Phys. Rev.* A **33** 3415
[9] Beysens D, Gbadamassi M and Boyer L 1979 *Phys. Rev. Lett* **43** 1253
Beysens D and Gbadamassi M 1979 *J. Phys. Lett.* **40** L565
[10] Onuki A and Kawasaki K 1980 *Prog. Theor. Phys.* **63** 122
Onuki A, Yamazaki K and Kawasaki K 1981 *Ann. Phys., NY* **131** 217
Imaeda T, Onuki A and Kawasaki K 1984 *Prog. Theor. Phys.* **71** 16
[11] Ruiz R and Nelson D R 1981 *Phys. Rev.* A **23** 3224
Ruiz R and Nelson D R 1981 *Phys. Rev.* A **24** 2727
Aronowitz A and and Nelson D R 1984 *Phys. Rev.* A **29** 2012
[12] Easwar N, Maher J V, Pine D J and Goldburg W I 1983 *Phys. Rev. Lett.* **51** 1272
Pine D J, Easwar N, Maher J V and Goldburg W I 1984 *Phys. Rev.* A **29** 308
Tong P, Goldburg W I, Stavans J and Onuki A 1989 *Phys. Rev. Lett.* **62** 2668
[13] Chan C K, Perrot F and Beysens D 1988 *Phys. Rev. Lett.* **61** 412
Perrot F, Chan C K and Beysens D 1989 *Europhys. Lett.* **9** 65
Chan C K 1990 *Chin. J. Phys.* **28** 75
Chan C K, Perrot F and Beysens D 1991 *Phys. Rev.* A **43** 1826
[14] Hnatich M 1990 *Teor. Mat. Fiz.* **83** 374
Antonov N V, Nalimov M Yu, Hnatich M and Horvath D 1998 *Int. J. Mod. Phys.* B **12** 1937
[15] Corberi F, Gonnella G and Lamura A 1999 *Phys. Rev. Lett.* **83** 4057
Corberi F, Gonnella G and Lamura A 2000 *Phys. Rev.* E **62** 8064
[16] Berthier L 2001 *Phys. Rev.* E **63** 051503
[17] Bray A J and Cavagna A 2000 *J. Phys. A: Math. Gen.* **33** L305
Cavagna A, Bray A J and Travasso R D M 2000 *Phys. Rev.* E **62** 4702
Bray A J, Cavagna A and Travasso R D M 2000 *Phys. Rev.* E **64** 012102
Bray A J, Cavagna A and Travasso R D M 2001 *Phys. Rev.* E **65** 016104
[18] Rapapa N P and Bray A J 1999 *Phys. Rev. Lett.* **83** 3856
Emmott C L and Bray A J 1999 *Phys. Rev.* E **59** 213
Rapapa N P 2000 *Phys. Rev.* E **61** 247
[19] Lacasta A M, Sancho J M and Sagués F 1995 *Phys. Rev. Lett.* **75** 1791
Berthier L, Barrat J-L and Kurchan J 2001 *Phys. Rev. Lett.* **86** 2014
Berti S, Boffetta G, Cencini M and Vulpiani A 2005 *Phys. Rev. Lett.* **95** 224501
[20] Antonov N V, Hnatich M and Honkonen J 2006 *J. Phys. A: Math. Gen.* **39** 7867
[21] Falkovich G, Gawędzki K and Vergassola M 2001 *Rev. Mod. Phys.* **73** 913
[22] Antonov N V 2006 *J. Phys. A: Math. Gen.* **39** 7825
[23] Avellaneda M and Majda A 1990 *Commun. Math. Phys.* **131** 381
Avellaneda M and Majda A 1992 *Commun. Math. Phys.* **146** 139
[24] Zhang Q and Glimm J 1992 *Commun. Math. Phys.* **146** 217
[25] Majda A 1993 *J. Stat. Phys.* **73** 515
Horntrop D and Majda A 1994 *J. Math. Sci. Univ. Tokyo* **1** 23
[26] Schmittman B and Zia R K P 1995 *Phase Transitions and Critical Phenomena* vol 17, ed C Domb and J Lebowitz (London: Academic) p 267
[27] Täuber U C 2003 *Adv. Solid State Phys.* **43** 659
[28] Janssen H-K and Täuber U C 2004 *Ann. Phys., NY* **315** 147
[29] Basu A and Frey E 2004 *Phys. Rev.* E **69** 015101

[30] Calabrese P and Gambassi A 2005 *J. Phys. A: Math. Gen.* **38** R133
[31] Schmittman B, Pruessner G and Janssen H-K 2006 *J. Phys. A: Math. Gen.* **39** 8183
[32] Mazenko G F, Valls O T and Zanetti M 1988 *Phys. Rev.* B **38** 520
Roland C and Grant M 1989 *Phys. Rev.* B **39** 11971
[33] Bray A J 1994 *Adv. Phys.* **43** 357
[34] Martin P C, Siggia E D and Rose H A 1973 *Phys. Rev.* A **8** 423
Janssen H K 1976 *Z. Phys.* B **23** 377
Bausch R, Janssen H K and Wagner H 1976 *Z. Phys.* B **24** 113
De Dominicis C 1976 *J. Phys. (Paris)* C **1** 247
[35] Larkin A I and Khmel'nitskii D E 1969 *Sov. Phys.—JETP* **29** 1123
Aharony A 1973 *Phys. Rev.* B **8** 3323
Brézin E and Zinn-Justin J 1976 *Phys. Rev.* B **13** 251
[36] Boyanovsky D and Cardy J L 1982 *Phys. Rev.* B **26** 154
Boyanovsky D and Cardy J L 1983 *Phys. Rev.* B **27** 6971 (erratum)
[37] Lawrie I D and Prudnikov V V 1984 *J. Phys. C: Solid State Phys.* **17** 1655
[38] Hwa T and Kardar M 1989 *Phys. Rev. Lett.* **62** 1813
Hwa T and Kardar M 1992 *Phys. Rev.* A **45** 7002
Tadic B 1998 *Phys. Rev.* E **58** 168
[39] Jeong H, Kahng B and Kim D 1996 *Phys. Rev. Lett.* **25** 5094
Kim H-J and Kim I-M 1998 *Phys. Rev.* E **58** 1144
[40] Mergulhão C Jr and Carneiro C E I 1999 *Phys. Rev.* B **59** 13954
Diehl H W and Spot M 2001 *J. Phys. A: Math. Gen.* **34** 9101
Diehl H W 2002 *Acta Phys. Slovaca* **52** 271
[41] Speer E R 1969 *Generalized Feynman Amplitudes* (Princeton, NJ: Princeton University Press)
[42] Adzhemyan L Ts, Honkonen J, Kompaniets M V and Vasil'ev A N 2005 *Phys. Rev.* E **71** 036305
[43] Antonov N V and Vasil'ev A N 1984 *Theor. Math. Phys.* **60** 671
[44] Prudnikov V V, Ivanov A V and Fedorenko A A 1997 *JETP Lett.* **66** 835
[45] Honkonen J and Nalimov M Yu 1996 *Z. Phys.* B **99** 297
Adzhemyan L Ts, Antonov N V and Vasil'ev A N 1999 *The Field Theoretic Renormalization Group in Fully Developed Turbulence* (London: Gordon and Breach) section 3.10
[46] Ohta T, Jasnow D and Kawasaki K 1982 *Phys. Rev. Lett.* **49** 1223
[47] Binosi D and Theussl L 2004 *Comput. Phys. Commun.* **161** 76